Lecture 20
Methods for solving the radiative transfer equation. Part 3: Discrete-ordinate method.

Objectives:
1. Discrete-ordinate method for the case of isotropic scattering.
2. Generalization of the discrete-ordinate method for inhomogeneous atmosphere.
3. Numerical implementation of the discrete-ordinate method: DISORT

Required reading:
L02: 6.2

Recommended reading
Thomas G.E. and K. Stamnes, Radiative transfer in the atmosphere and ocean, 2000, Chapter 8.1-8.10

1. Discrete-ordinate method for the case of isotropic scattering.

- A discrete-ordinate method has been developed by Chandrasekhar in about 1950 (see Chandrasekhar S., Radiative transfer, 1960, Dover Publications).

Recall the radiative transfer equation for azimuthally independent diffuse intensity:

$$
\mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - \frac{\omega_0}{2} \int_{-1}^{1} I(\tau, \mu') P(\mu', \mu) d\mu' - \frac{\omega_0}{4\pi} F_0 P(\mu, -\mu_0) \exp(-\tau / \mu_0)
$$

For isotropic scattering, the scattering phase function is 1. Hence we have

$$
\mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - \frac{\omega_0}{2} \int_{-1}^{1} I(\tau, \mu') d\mu' - \frac{\omega_0 F_0}{4\pi} \exp(-\tau / \mu_0) \quad [20.1]
$$

Let’s apply the Gauss formula to replace the integral in Eq.[20.1]

$$
\mu_i \frac{dI(\tau, \mu_i)}{d\tau} = I(\tau, \mu_i) - \frac{\omega_0}{2} \sum_{j=-n}^{n} a_j I(\tau, \mu_j) - \frac{\omega_0 F_0}{4\pi} \exp(-\tau / \mu_0) \quad [20.2]
$$

where \(i=-n, \ldots, n\) (2n terms) and \(a_j\) are the Gaussian weights (constants) and \(\mu_i\) are quadrature angles (or points).
Eq.[20.2] is a system of 2n inhomogeneous differential equations:

Solution of Eq.[20.2] = general solution + particular solution

where the general solution is a solution of the homogeneous part of the Eq.[20.2]

Denoting \( I_i = I_i (\tau, \mu_i) \), the general solution of Eq.[20.2] can be found as

\[
I_i = g_i \exp(-k \tau)
\]  

[20.3]

Inserting Eq.[20.3] into Eq.[20.2], we obtain

\[
g_i (1 + \mu_i k) = \frac{\omega_0}{2} \sum_{j=-n}^{n} a_j g_j
\]  

[20.4]

We can find \( g_i \) in the form

\[
g_i = L_i (1 + \mu_i k)
\]

where L is a constant to be determined. Substituting this expression for \( g_i \) in Eq.[20.4], we have

\[
1 = \frac{\omega_0}{2} \sum_{j=-n}^{n} \frac{a_j}{1 + \mu_i k} = \omega_0 \sum_{j=1}^{n} \frac{a_j}{1 - \mu_j^2 k^2}
\]  

[20.5]

Eq.[20.5] gives 2n solutions for +/-kj (j=1,…,n).

Thus general solution is

\[
I_i = \sum_j \frac{L_j}{1 + \mu_j k_j} \exp(-k_j \tau)
\]  

[20.6]

where \( L_j \) are constants.

The particular solution can be found as

\[
I_i = \frac{\omega_0 F_0}{4\pi} h_i \exp(-\tau / \mu_0)
\]  

[20.7]

where \( h_i \) are constants.

Inserting Eq.[20.7] into Eq.[20.2], we have

\[
h_i (1 + \mu_i / \mu_0) = \frac{\omega_0}{2} \sum_{j=-n}^{n} a_j h_j + 1
\]  

[20.8]

From Eq.[20.8], \( h_i \) is found as

\[
h_i = \gamma / (1 + \mu_i / \mu_0)
\]
where $\gamma$ is determined from

$$
\gamma = 1 / \left(1 - \frac{\omega_0}{2} \sum_{j=1}^{N} a_j \left(1 - \mu_j^2 / \mu_0^2\right)\right) \quad [20.9]
$$

Adding the general solution Eq.[20.6] and the particular solution Eq.[20.7], we have the solution

$$
I_j = \sum_j \frac{L_j}{1 + \mu_j k_j} \exp(-k_j \tau) + \frac{\omega_0 F_0 \gamma}{4\pi (1 + \mu / \mu_0)} \exp(-\tau / \mu_0) \quad [20.10]
$$

where $L_j$ are constants to be determined from the boundary conditions.

**H-function** has been introduced by Chandrasekhar as

$$
H(\mu) = \frac{1}{\mu_1 \cdots \mu_n} \prod_{j=1}^{n} \frac{1}{(1 + k_j \mu)} \quad [20.11]
$$

One can express $\gamma$ in the H-function that Eq.[20.10] becomes

$$
I_j = \sum_j \frac{L_j}{1 + \mu_j k_j} \exp(-k_j \tau) + \frac{\omega_0 F_0 H(\mu_0) H(-\mu_0)}{4\pi (1 + \mu / \mu_0)} \exp(-\tau / \mu_0) \quad [20.12]
$$

Eq.[20.12] gives a simple solution for the semi-infinite isotropic atmosphere (see L02:6.2.2)

$$
I^\uparrow(0, \mu) = \frac{1}{4\pi} \frac{\omega_0 F_0 \mu_0}{\mu + \mu_0} H(\mu_0) H(\mu) \quad [20.13]
$$

2. **Generalization of the discrete-ordinate method for inhomogeneous atmosphere.**

Let’s consider the atmosphere with non-isotropic scattering.

We can expand the diffuse intensity in the cosine series

$$
I(\tau, \mu, \varphi) = \sum_{m=0}^{N} I^m(\tau, \mu) \cos(m(\varphi_0 - \varphi))
$$
So we need to solve
\[
\mu \frac{d I^m_i}{d \tau} = I^m_i - (1 + \delta_{0,m}) \frac{\omega_0}{4} \sum_{l=m}^{N} \sigma_i^m P_i^m(\mu) \int_{-1}^{1} P_{l}^m(\mu') I^m(\tau, \mu') d \mu' - \frac{\omega_0}{4\pi} \sum_{l=m}^{N} \sigma_i^m P_i^m(\mu) P_l^m(-\mu_0) F_0 \exp(-\tau / \mu_0)
\]

The **general solution** may be written
\[
I^m_i(\tau, \mu_i) = \sum_{j=-n}^{n} L_j^m \phi_j^m(\mu_j) \exp(-k_j^m \tau)
\]

\(\phi_j^m, k_j^m, L_j^m\) are coefficients to be determined.

The **particular solution** may be written
\[
I_p^m(\tau, \mu_i) = Z^m(\mu_i) \exp(-\tau / \mu_0)
\]

\(Z^m(\mu_i)\) is a function
\[
Z^m(\mu_i) = \frac{1}{4\pi} \omega_0 F_0 P_0^m(-\mu_0) \frac{H^m(\mu_0) H^m(-\mu_0)}{1 + \mu_i / \mu_0} \sum_{l=0}^{N} \sigma_i^m \xi_j^m \frac{1}{\mu_0} P_l^m(\mu_i)
\]

The **complete solution** of the radiative transfer is
\[
I^m_i(\tau, \mu_i) = \sum_{j=-n}^{n} L_j^m \phi_j^m(\mu_j) \exp(-k_j^m \tau) + Z^m(\mu_i) \exp(-\tau / \mu_0) \quad [20.14]
\]

\(i=-n,\ldots, n\)

Let’s generalize the **complete solution** Eq.[20.14] of the radiative transfer for the inhomogeneous atmosphere. The atmosphere can be divided into the N homogeneous layers, each is characterized by a single scattering albedo, phase function, and optical depth.

**NOTE:** If a layer has gases, aerosols and/or clouds, one needs to calculate the effective optical properties of this layer.
For \( l \)-th layer, we can write the solution using Eq.[20.14]. To simplify notations, let’s consider the azimuthal independent case (i.e., \( m=0 \)), so we have

\[
I^l(\tau, \mu_i) = \sum_{j=-n}^{n} L_j^l \phi_j^l(\mu_j) \exp(-k_j^l \tau) + Z^l(\mu_i) \exp(-\tau / \mu_0) \tag{20.15}
\]

Now, we need to match the boundary and continuity conditions between layers.

**At the top of the atmosphere (TOA):** no downward diffuse intensity

\[
I^{i=1}(0, -\mu_i) = 0 \tag{20.16}
\]

**At the layer’s boundary:** upward and downward intensities must be continuous

\[
I^l(\tau_i, \mu_i) = I^{i+1}(\tau_i, \mu_i) \tag{20.17}
\]

**At the bottom of the atmosphere** (assuming the Lamdertian surface):

\[
I^{i=N}(\tau_n, \mu_i) = \frac{\tau_{\text{sur}}}{\pi} [F^l(\tau_n) + \mu_0 F_0 \exp(-\tau_n / \mu_0)] \tag{20.18}
\]

Eqs.[20.16]-[20.18] provide necessary equations to find the unknown coefficients.

### 3. Numerical implementation of the discrete-ordinate method: DISORT

DISORT is a FORTRAN numerical code based on the discrete-ordinate method developed by Stamnes, Wiscombe et al. DISORT is openly available and has a good user-guide.

1) DISORT applies to the inhomogeneous nonithothermal plane-parallel atmosphere.
2) A user may set-up any numbers of the plane-parallel layers.
3) Each layer must be characterized by the effective optical depth, single scattering albedo and asymmetry parameter if the Henyey-Greenstein phase function is used.
4) A user may use any phase function by providing the Legendre polynomial expansion coefficients.
5) A user selects a number of streams (keeping in mind that the computation time varies as \( n^3 \)).
6) A key problem is to obtain a solution for fluxes for strongly forward-peaked scattering.
7) DISORT allows predicting the intensity as a function of the direction and position at any point in the atmosphere (i.e., not only at the boundaries of the layers).