Lecture 20

Methods for solving the radiative transfer equation.

Part 3: Discrete-ordinate method.

Objectives:
1. Discrete-ordinate method for the case of isotropic scattering.
2. Generalization of the discrete-ordinate method for inhomogeneous atmosphere.
3. Numerical implementation of the discrete-ordinate method: DISORT
4. Examples: Modeling with DISORT.

Required reading:
L02: 6.2

Additional/Advanced reading:
Thomas G.E. and K. Stamnes, Radiative transfer in the atmosphere and ocean, 2000, Chapter 8.1-8.10

1. Discrete-ordinate method for the case of isotropic scattering.

• A discrete-ordinate method has been developed by Chandrasekhar in about 1950 (see Chandrasekhar S., Radiative transfer, 1960, Dover Publications).

Recall the radiative transfer equation (Eq.17.7) for azimuthally independent diffuse intensity:

$$\mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - \frac{\omega_0}{2} \int_{-1}^{1} I(\tau, \mu') P(\mu, \mu') d\mu' - \frac{\omega_0 F_0}{4\pi} P(\mu, -\mu_0) \exp(-\tau / \mu_0)$$

For isotropic scattering, the scattering phase function is 1. Hence we have

$$\mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - \frac{\omega_0}{2} \int_{-1}^{1} I(\tau, \mu') d\mu' - \frac{\omega_0 F_0}{4\pi} \exp(-\tau / \mu_0) \tag{20.1}$$

Let’s apply the Gaussian quadratures to replace the integral in Eq.[20.1]

$$\mu_i \frac{dI(\tau, \mu_i)}{d\tau} = I(\tau, \mu_i) - \frac{\omega_0}{2} \sum_{j=-n}^{n} a_j I(\tau, \mu_j) - \frac{\omega_0 F_0}{4\pi} \exp(-\tau / \mu_0) \tag{20.2}$$

Inhomogeneous part
where \( i = -n, \ldots, n \) (2n terms) and \( a_j \) are the Gaussian weights (constants) and \( \mu_j \) are quadrature angles (or points).

Eq.[20.2] is a system of 2n inhomogeneous differential equations:

**Solution of Eq.[20.2] = general solution + particular solution**

where the general solution is a solution of the homogeneous part of the Eq.[20.2]

Denoting \( I_i = I_i (\tau, \mu_i) \), the general solution of Eq.[20.2] can be found as

\[
I_i = g_i \exp(-k \tau)
\]  

[20.3]

Inserting Eq.[20.3] into Eq.[20.2], we obtain

\[
g_i (1 + \mu_i k) = \frac{\alpha_0}{2} \sum_{j=-n}^{n} a_j g_j
\]  

[20.4]

We can find \( g_i \) in the form

\[
g_i = L/(1 + \mu_i k)
\]

where \( L \) is a constant to be determined. Substituting this expression for \( g_i \) in Eq.[20.4], we have

\[
1 = \frac{\alpha_0}{2} \sum_{j=-n}^{n} \frac{a_j}{1 + \mu_i k} = \alpha_0 \sum_{j=1}^{n} \frac{a_j}{1 - \mu_j^2 k^2}
\]  

[20.5]

Eq.[20.5] gives 2n solutions for +/-\( k_j \) (j=1,..,n).

Thus general solution is

\[
I_i = \sum_j \frac{L_j}{1 + \mu_j k_j} \exp(-k_j \tau)
\]  

[20.6]

where \( L_j \) are constants.

The particular solution can be found as

\[
I_i = \frac{\alpha_0 F_0}{4 \pi} h_i \exp(-\tau / \mu_0)
\]  

[20.7]

where \( h_i \) are constants.

Inserting Eq.[20.7] into Eq.[20.2], we have

\[
h_i (1 + \mu_i / \mu_0) = \frac{\alpha_0}{2} \sum_{j=-n}^{n} a_j h_j + 1
\]  

[20.8]
From Eq.[20.8], $h_i$ is found as

$$h_i = \gamma / (1 + \mu_i / \mu_o)$$

where $\gamma$ is determined from

$$\gamma = 1 / \{1 - \frac{\omega_0}{2} \sum_{j=1}^{n} a_j / (1 - \mu_j^2 / \mu_o^2)\}$$  \[20.9\]

Adding the general solution Eq.[20.6] and the particular solution Eq.[20.7], we have

$$I_i = \sum_j \frac{L_j}{1 + \mu_i k_j} \exp(-k_j \tau) + \frac{\omega_0 F_0 \gamma}{4 \pi (1 + \mu_i / \mu_o)} \exp(-\tau / \mu_o)$$  \[20.10\]

where $L_j$ are constants to be determined from the boundary conditions.

**H-function** has been introduced by Chandrasekhar as

$$H(\mu) = \frac{1}{\mu_1 \ldots \mu_n} \prod_{j=1}^{n} (\mu + \mu_j) \prod_{j=1}^{n} (1 + k_j \mu)$$  \[20.11\]

Expressing $\gamma$ in the H-function, Eq.[20.10] becomes

$$I_i = \sum_j \frac{L_j}{1 + \mu_i k_j} \exp(-k_j \tau) + \frac{\omega_0 F_0 H(\mu_o) H(-\mu_o)}{4 \pi (1 + \mu_i / \mu_o)} \exp(-\tau / \mu_o)$$  \[20.12\]

Eq.[20.12] gives a simple solution for the semi-infinite isotropic atmosphere (see L02:6.2.2)

$$I^+(0, \mu) = \frac{1}{4 \pi} \omega_0 F_0 \frac{\mu_0}{\mu + \mu_0} H(\mu_o) H(\mu)$$  \[20.13\]

2. Generalization of the discrete-ordinate method for an inhomogeneous atmosphere.

Let’s consider the atmosphere with non-isotropic scattering.

We can expand the diffuse intensity in the cosine series

$$I(\tau, \mu, \varphi) = \sum_{m=0}^{N} I^m(\tau, \mu) \cos( m(\varphi_0 - \varphi))$$
So we need to solve

\[
\mu \frac{dI^m(\tau, \mu)}{d\tau} = I^m(\tau, \mu) - (1 + \delta_{0,m}) \frac{\omega_0}{4} \sum_{l=m}^N \sigma_i^m P_i^m(\mu) \int_{-1}^1 P_i^m(\mu')I^m(\tau, \mu')d\mu' - \\
\frac{\omega_0}{4\pi} \sum_{l=m}^N \sigma_i^m P_i^m(\mu)P_i^m(-\mu)F_0 \exp(-\tau/\mu_0)
\]

The general solution may be written

\[
I^m(\tau, \mu_i) = \sum_{j=-n}^n L_j^m \phi_j^m(\mu_i) \exp(-k_j^m\tau)
\]

\(\phi_j^m, k_j^m, L_j^m\) are coefficients to be determined.

The particular solution may be written

\[
I_p^m(\tau, \mu_i) = Z^m(\mu_i) \exp(-\tau/\mu_0)
\]

\(Z^m(\mu_i)\) is a function

\[
Z^m(\mu_i) = \frac{1}{4\pi} \omega_0 F_0 P_m^m(-\mu_0) \frac{H^m(\mu_0)H^m(-\mu_0)}{1 + \mu_i/\mu_0} \sum_{l=0}^N \omega_i^m \xi_i^m \frac{1}{\mu_0} P_i^m(\mu_i)
\]

The complete solution of the radiative transfer is

\[
I^m(\tau, \mu_i) = \sum_{j=-n}^n L_j^m \phi_j^m(\mu_i) \exp(-k_j^m\tau) + Z^m(\mu_i) \exp(-\tau/\mu_0) \tag{20.14}
\]

Let’s generalize the complete solution Eq.[20.14] of the radiative transfer for the inhomogeneous atmosphere. The atmosphere can be divided into the N homogeneous layers, each is characterized by a single scattering albedo, phase function, and optical depth.

**NOTE:** If an atmospheric layer has gases, aerosols and/or clouds, one needs to calculate the effective optical properties of this layer (see Eqs.[14.31]-[14.34]).
For $l$-th layer, we can write the solution using Eq.[20.14]. To simplify notations, let’s consider the azimuthal independent case (i.e., $m=0$), so we have

$$I^l(\tau, \mu_i) = \sum_{j=-n}^{n} L^l_j \phi^l_j(\mu_j) \exp(-k^l_j \tau) + Z^l(\mu_i) \exp(-\tau / \mu_0)$$ \hspace{1cm} [20.15]

Now, we need to match the boundary and continuity conditions between layers.

**At the top of the atmosphere (TOA):** no downward diffuse intensity

$$I^{l+1}(0, -\mu_i) = 0$$ \hspace{1cm} [20.16]

**At the layer’s boundary:** upward and downward intensities must be continuous

$$I^l(\tau_i, \mu_i) = I^{l+1}(\tau_i, \mu_i)$$ \hspace{1cm} [20.17]

**At the bottom of the atmosphere** (assuming the Lambertian surface):

$$I^{l=N}(\tau_N, \mu_i) = \frac{r\text{\,\,sur}}{\pi} [F^l(\tau_N) + \mu_0 F_0 \exp(-\tau_N / \mu_0)]$$ \hspace{1cm} [20.18]

Eqs.[20.16]-[20.18] provide necessary equations to find the unknown coefficients.

### 3. Numerical implementation of the discrete-ordinate method: DISORT

DISORT is a FORTRAN numerical code based on the discrete-ordinate method developed by Stamnes, Wiscombe et al.

DISORT is openly available and has a good user-guide.

**Some features:**

1) DISORT applies to the inhomogeneous nonithothermal plane-parallel atmosphere.
2) A user may set-up any numbers of the plane-parallel layers.
3) Each layer must be characterized by the effective optical depth, single scattering albedo and asymmetry parameter if the Henyey-Greenstein phase function is used.
4) A user may use any phase function by providing the Legendre polynomial expansion coefficients.
5) A user selects a number of streams (keeping in mind that the computation time varies as $n^3$).
6) A key problem is to obtain a solution for fluxes for strongly forward-peaked scattering.
7) DISORT allows predicting the intensity as a function of the direction and position at any point in the atmosphere (i.e., not only at the boundaries of the layers).