Lecture 9

Terrestrial infrared radiative processes. Part 2:

K-distribution approximation.

Objectives:
2. Correlated k-distribution approximation (CKD).

Required reading:
L02:4.3

Additional reading:


The KD method is developed to compute the spectral transmittance (hence the spectral intensity or spectral fluxes) based on grouping of gaseous absorption coefficients.

NOTE: The k-distribution approach was proposed by Ambartzumian in 30th as an alternative to the computationally expensive line-by-line methods.

- KD method benefits from the fact that the same value of $k_\nu$ is encountered many times over a given spectral interval => thus to eliminate the redundancy, one can group the values of $k$ and perform the transmittance calculation only once for a given value of $k$. 
**Strategy:**

Consider a **homogeneous** atmospheric layer. Spectral transmission is (see Eq.[8.6], Lecture 8)

\[
T_r(u) = \frac{1}{\Delta \nu} \int \exp(-k, u) d\nu
\]

In a homogeneous atmospheric layer, spectral transmittance is independent of the ordering of \( k \) in a given spectral range, i.e., the order in which the wavenumbers are summed does not matter => so sum them from low to high \( k \)

Thus, we want to replace the integration over the wavenumber by an integration over \( k \). It can be done by introducing a **normalized probability distribution function** for \( k_\nu \)

\[
T_r(u) = \frac{1}{\Delta \nu} \int \exp(-k, u) d\nu = \int_0^\infty \exp(-ku)f(k)dk \tag{9.1}
\]

where we set \( \int_0^\infty f(k)dk = 1 \)

\( f(k) \) is the fraction of the spectral band with absorption coefficient \( k \to k+dk \)

\( f(k) \) is a smooth function

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**Figure 9.1 (a)** A schematic of absorption line spectra at two different pressure. **(b)** The two probability density function \( f(k) \) associated with (a). The shaded area shows the strongest absorption.
The cumulative probability function can be defined as

\[ g(k) = \int_0^k f(k) \, dk \]  

and \( g(0) = 0; \quad g(\infty) = 1 \) and \( dg(k) = f(k) \, dk \).

\( g(k) \) is the fraction of the spectrum with absorption coefficient below \( k \)

**NOTE:** By definition, \( g(k) \) is a monotonically increasing and smooth function in \( k \)-space, Therefore, \( k(g) \), as an inverse function of \( g(k) \), is a smooth function in \( g \)-space.

Therefore, the spectral transmittance can be written as

\[ T_v(u) = \frac{1}{\Delta v} \int_{\Delta v} \exp(-k,u) \, du = \int_0^\infty \exp(-ku) f(k) \, dk = \int_0^1 \exp(-k(g)u) dg \]  

**Figure 9.2** (a) Absorption coefficient \( k_\nu \) (in \( \text{cm}^{-1} \text{ atm}^{-1} \)) as a function of wavenumber in 9.6 \( \mu \text{m} \) ozone band (resolution of 0.05 cm\(^{-1} \), \( p=30 \text{ mb}, T=200K \)). (b) The probability density function \( f(k) \) of the absorption coefficient. (c) The cumulative probability distribution function as a function of \( k \). (d) Same as (c) but \( k \) vs. \( g \).
Because both $g(k)$ and $k(g)$ are smooth functions, the above integral can be calculated by a finite sum as

$$T_p(u) = \int_0^1 \exp(-k(g)u)dg \approx \sum_{i=1}^N \exp(-k(g_i)u)\Delta g_i = \Delta g_1 e^{-k_{1}u} + \Delta g_2 e^{-k_{2}u} + \cdots + \Delta g_N e^{-k_{N}u}$$

where $\Delta g_i$ is the quadrature weight.

Thus the **KD method** allows calculating the spectral transmittance as a finite weighted sum of exponent in g-space, replacing the tedious wavenumber integration.

**Numerical realization of KD:**

*(see illustration below)*

Consider a spectral interval $\Delta \nu$ that contains numerous absorption lines.

Let’s divide it into $N$ intervals of $\Delta \nu_j$, $j = 1, 2, 3 \ldots N$

The probability distribution function can be written as

$$f(k) = \frac{1}{\Delta \nu} \frac{d\nu}{dk} = \frac{1}{\Delta \nu} \sum_j \frac{\Delta \nu_j}{\Delta k}$$

where $\Delta \nu_j$ is the subinterval of $\Delta \nu$ where $k$ is a monotonic function of $\nu$.

Then the cumulative probability is

$$g(k) = \frac{1}{\Delta \nu} \sum_j \int_0^k \frac{\Delta \nu_j}{\Delta k} dk' = \frac{1}{\Delta \nu} \sum_j \int_0^k \Delta \nu_j(k) = \frac{n(0,k)}{N}$$

where $n(0,k)$ is the number of computational points that contribute to $k$ cumulatively.
Figure 9.3 How to calculate the absorption coefficient in g-space from the known absorption coefficient in the wavenumber domain. Solid line gives absorption coefficient as a function of $\nu$. Numbers on the right side are the data points in each $\Delta k$ interval (total number $N=35$).

Thus by definition, $g(j\Delta k)=n(0, j\Delta k)/N$

**CKD** is the extension of KD for an inhomogeneous atmosphere.

Each pressure and temperature along the path has a unique \( k_v \) spectrum.

**CKD method** sorts each \( k_v(p, T) \) spectrum independently to make k-distributions \( k(g, p, T) \) for each \( p \) and \( T \).

NOTE: In practice, discrete k-distributions \( k_j(p_l, T_m) \) are made for a set of pressures \( p_l \) and temperatures \( T_m \) and interpolated in between to any \( p \) and \( T \).

- **Overlap of gases in spectral band:**

  What do we do about multiple gases absorbing in one spectral band?

**Overlap method #1:** assume that absorption spectra are independent.

\[
\int_{\Delta \nu} T_v^{(1)} T_v^{(2)} d\nu = T_{g}^{(1)} T_{g}^{(2)} = \sum_{i=1}^{N} \exp(-k_i(g)u_1)\Delta g_{1,i} \sum_{j}^{M} \exp(-k(g)u_2)\Delta g_{2,j}
\]

\[
T_g(u_1, u_2) = \sum_{i=1}^{N} \Delta g_{1,i} \sum_{j}^{M} \Delta g_{2,j} \exp(-\tau_{mn})
\]

where \( \tau_{mn} = k_{1n} u_1 + k_{2n} u_2 \),

thus we have \( M \times N \) terms.

**Overlap method #2:** introduce mixing ratio as an additional factor, so \( k(g, p, T, q) \)

- **An example of correlated k-distribution models**


Divides shortwave into 6 bands with total 54 \( k \)’s and longwave into 12 bands with total of 121 \( k \)’s.
<table>
<thead>
<tr>
<th>Band</th>
<th>Region (cm$^{-1}$)</th>
<th>$N$ g's</th>
<th>Gases</th>
<th>Solar Flux (W/m$^2$)</th>
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<tr>
<td>1</td>
<td>50000-14500</td>
<td>10</td>
<td>O3</td>
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<td>2</td>
<td>14500-7700</td>
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<td>H2O</td>
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<td>2850-2500</td>
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<td>2200-1900</td>
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<tr>
<td>8</td>
<td>1900-1700</td>
<td>3</td>
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<tr>
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<td>1700-1400</td>
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<td>1400-1250</td>
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