The Delta-Eddington Approximation for Radiative Flux Transfer

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ABSTRACT

This paper presents a rapid yet accurate method, the "delta-Eddington" approximation, for calculating monochromatic radiative fluxes in an absorbing-scattering atmosphere. By combining a Dirac delta function and a two-term approximation, it overcomes the poor accuracy of the Eddington approximation for highly asymmetric phase functions. The fraction of scattering into the truncated forward peak is taken proportional to the square of the phase function asymmetry factor, which distinguishes the delta-Eddington approximation from others of similar nature. Comparisons of delta-Eddington albedos, transmissivities and absorptivities with more exact calculations reveal typical differences of 0–0.02 and maximum differences of 0.15 over wide ranges of optical depth, sun angle, surface albedo, single-scattering albedo and phase function asymmetry factor. Delta-Eddington fluxes are in error, on the average, by no more than 0.5%, and at the maximum by no more than 2% of the incident flux. This computationally fast and accurate approximation is potentially of utility in applications such as general circulation and climate modeling.

1. Introduction

General circulation models and climate models, as well as certain agricultural and engineering applications, require the frequent evaluation of solar fluxes and heating rates at many locations. The most serious challenge facing radiative transfer theory in such applications is to devise simple and computationally fast analytic approximations having adequate fidelity to the more exact numerical solutions (which in turn incorporate approximations such as horizontal homogeneity). This problem is complicated by the wide ranges of values that radiative parameters may take on in realistic atmospheres. Many approximate methods are valid only for severely restricted ranges of one or more of these parameters.

The most simple and popular approximations to the complicated integro-differential equation of radiative transfer are the Eddington and the two-stream, and variants thereof. They both represent the angular dependence of the radiation intensity in terms of just two functions of optical depth, for which a pair of ordinary differential equations is derived. For homogeneous layers these differential equations have constant coefficients and, therefore, simple exponential solutions for fluxes. These solutions are framed in terms of the optical depth, the albedo for single scattering, and one or two moments of the scattering phase function.

Simple approximations, like the Eddington, are often incapable of coping with the highly asymmetric phase functions typical of particulate scattering. Fritz (1954, 1958), Irvine (1965) and Joseph (1968, 1970, 1971) noted that it is reasonable to replace the large forward peak in such phase functions by a Dirac delta function. They assumed that half of the scattered radiation goes into the forward peak. Fritz and Irvine then took the remaining scattering to be completely isotropic, while Joseph assumed isotropy separately in each hemisphere. Weinman (1968), Hansen (1969) and Potter (1970) truncated the peak by extrapolating the phase function from angles outside the peak; the remainder of the phase function was left intact. Excellent accuracy for albedo was achieved by this approximation except for nearly grazing angles. Wang (1972) approximated the untruncated part of the phase function by an Eddington-type phase function, but he chose the fraction scattered into the forward peak in such a way as to conserve the integral of the original phase function from 90° to 180°. He also uses the exponential kernel rather than the Eddington approximation and centers his interest on the albedo of semi-infinite atmospheres, for astrophysical applications.

2. The delta-Eddington approximation

We approximate the phase function by a Dirac delta function forward scatter peak and a two-term

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expansion of the phase function

$$P(\cos\theta)\approx P_{\delta-\mathrm{Edd}}(\cos\theta)\equiv 2f\delta(1-\cos\theta)+(1-f)(1+3g'\cos\theta), \quad (1)$$

where f is the fractional scattering into the forward peak and g' is the asymmetry factor of the truncated phase function. It is easily shown that the delta-Eddington phase function is correctly normalized, i.e.,

$$\int_{4\pi} P_{\delta-\mathrm{Edd}}(\cos\theta)\frac{d\Omega}{4\pi}=\frac{1}{2}\int_{-1}^{1} P_{\delta-\mathrm{Edd}}(\mu)d\mu=f+(1-f)=1.$$

We further require $P_{\delta-\mathrm{Edd}}$ to have the same asymmetry factor (g) as the original phase function:

$$g=\int_{4\pi}\cos\theta P_{\delta-\mathrm{Edd}}(\cos\theta)\frac{d\Omega}{4\pi}=f+(1-f)g', \quad (2a)$$

from which g' is determined as

$$g'=\frac{g-f}{1-f}. \quad (2b)$$

Finally, to determine f we require that the second moment of $P_{\delta-\mathrm{Edd}}$,

$$\int_{4\pi} P_2(\cos\theta)P_{\delta-\mathrm{Edd}}(\cos\theta)\frac{d\Omega}{4\pi}=f, \quad (3)$$

be identical to the second moment of the original phase function P which is g^2 if P is approximated by the Henyey–Greenstein phase function

$$P_{\mathrm{H\text{-}G}}(\cos\theta)=\sum_{l=0}^{\infty}(2l+1)g^l P_l(\cos\theta). \quad (4)$$

(van de Hulst, 1968). Both van de Hulst (1968) and Hansen (1969) show that, for flux computations, Henyey–Greenstein phase functions may be used in place of the more realistic ones from Mie theory. Thus we require

$$f=g^2 \quad (5a)$$

and therefore

$$g'=\frac{g}{1+g} \quad (5b)$$

from Eq. (2b).[3] Note that Eq. (5b) implies $0\leqslant g'\leqslant 0.5$ when $0\leqslant g\leqslant 1$, so the two-term part of $P_{\delta-\mathrm{Edd}}$ [Eq. (1)] applies to a range of g' for which the Eddington approximation is demonstrably accurate (Wiscombe *et al.*, 1976).

To estimate the difference between the delta-Eddington and Henyey–Greenstein phase functions, we expand the delta function in Eq. (1) in a Legendre polynomial series (Morse and Feshbach, 1953):

$$P_{\delta-\mathrm{Edd}}(\cos\theta)=f\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)+(1-f)[1+3g'P_1(\cos\theta)]$$
$$=1+3g\cos\theta+\sum_{l=2}^{\infty}(2l+1)g^2P_l(\cos\theta), \quad (6)$$

where Eqs. (5a) and (5b) have been used to replace f and g'. The delta-Eddington phase function therefore agrees with that of Henyey–Greenstein [Eq. (4)] out to three terms, and their difference is

$$P_{\delta-\mathrm{Edd}}(\cos\theta)-P_{\mathrm{H\text{-}G}}(\cos\theta)=\sum_{l=3}^{\infty}(2l+1)(g^2-g^l)P_l(\cos\theta). \quad (7)$$

Because the terms on the right-hand side of this equation approach zero as $g\rightarrow 1$, the delta-Eddington error tends to grow smaller rather than larger as $g\rightarrow 1$.

We now investigate the consequences of using the delta-Eddington approximate phase function in the radiative transfer equation. Since our interest is in fluxes only, it is sufficient to deal with the azimuthally averaged form of that equation, viz.,

$$\mu\frac{\partial I}{\partial\tau}+I=\frac{\omega}{2}\int_{-1}^{1}\bar{P}(\mu,\mu')I(\tau,\mu')d\mu', \quad (8)$$

where I is the azimuthally-averaged intensity and

$$\bar{P}(\mu,\mu')=\frac{1}{2\pi}\int_0^{2\pi}P_{\delta-\mathrm{Edd}}(\cos\theta)d\phi$$
$$=2f\delta(\mu-\mu')+(1-f)(1+3g'\mu\mu'). \quad (9)$$

Although thermal emission sources may be included in Eq. (8) and in our approximation, we omit them for reasons of simplicity. In Eq. (9) we have used the fact that when θ is the angle between the incident and scattering directions $\mathbf{\Omega}=(\mu,\phi)$ and $\mathbf{\Omega}'=(\mu',\phi')$ (in the usual notation), then

$$\cos\theta=\mu\mu'+(1-\mu^2)^{\frac{1}{2}}(1-\mu'^2)^{\frac{1}{2}}\cos(\phi-\phi'). \quad (10)$$

Also we have written the delta-function in $P_{\delta-\mathrm{Edd}}$ in the form

$$\delta(1-\cos\theta)=2\pi\delta(\mu-\mu')\delta(\phi-\phi'). \quad (11)$$

[3] Dual Henyey-Greenstein functions have sometimes been introduced to approximate phase functions peaked in the forward and backward directions. If $P_{\mathrm{D-H-G}}(\cos\theta)=b\;P_{\mathrm{H-G}}(g_1,\cos\theta)+(1-b)P_{\mathrm{H-G}}(g_2,\cos\theta)$, it follows that $g'=[bg_1(1-g_1)+(1-b)g_2\times(1-g_2)]/[1-bg_1^2-(1-b)g_2^2]$ and $f=bg_1^2+(1-b)g_2^2$. Although results will not be presented in this paper, agreement between fluxes computed rigorously and by the delta-Eddington approximation for $bg_1+(1-b)g_2>0$ is comparable to that shown in Figs. 1–3 of this study.

Inserting Eq. (9) into the transfer equation leads to

$$\mu\frac{\partial I}{\partial\tau'}+I=\frac{\omega'}{2}\int_{-1}^{1}(1+3g'\mu\mu')I(\tau',\mu')d\mu', \tag{12}$$

which is similar to the original transfer equation except that the variables have been transformed as follows:

$$\tau'=(1-\omega f)\tau \tag{13}$$

$$\omega'=\frac{(1-f)\omega}{1-\omega f}. \tag{14}$$

Making the usual diffuse-direct transformation (where πF_0 is the monodirectional flux at $\tau'=0$ incident at zenith angle $\cos^{-1}\mu_0$),

$$I=\begin{cases} i+\frac{1}{2}F_0\delta(\mu-\mu_0)\exp(-\tau'/\mu_0), & 0<\mu\leqslant 1\\ i, & -1\leqslant\mu<0\end{cases} \tag{15}$$

brings the transfer equation into the form

$$\mu\frac{\partial i}{\partial\tau'}+i=\omega'(i_0+g'\mu i_1) +\tfrac{1}{4}F_0\omega'(1+3g'\mu_0\mu)\exp(-\tau'/\mu_0), \tag{16a}$$

where

$$i_n(\tau')=\tfrac{1}{2}(2n+1)\int_{-1}^{1}\mu^n i(\tau',\mu)d\mu,\quad n=0,\ 1. \tag{16b}$$

If we now take the zeroth and first moments of the transfer equation by applying the operators $\int_{-1}^{1}d\mu$ and $\int_{-1}^{1}\mu d\mu$, and make the Eddington assumption that $i=i_0+\mu i_1$, we arrive at a pair of differential equations for i_0 and i_1. The solution to these equations is given by Eqs. (12) or (16) of Shettle and Weinman (1970), with g', τ', ω' replacing g, τ, ω.

The delta-Eddington approximation is thus equivalent to the Eddington approximation with transformed parameters g', τ', ω' [Eqs. (5b), (13), (14)]. van de Hulst (1974) has developed several such similarity transformations in radiative transfer.

These transformations are

$$\frac{\tau}{\tau'}=\frac{1-\omega'}{1-\omega}=\frac{\omega'(1-g')}{\omega(1-g)} \tag{17a}$$

and render it feasible to transform a problem with a strongly anisotropic phase function to one with $g'<g$. Eq. (17a) is satisfied by Eqs. (2b), (13) and (14). By further making the approximation outlined by Eqs. (3) and (4), we are led deductively to $f=g^2$ and also satisfy van de Hulst's empirical approximate similarity relations

$$\frac{1-\omega'}{1-\omega'g'}=\frac{1-\omega}{1-\omega g}, \tag{17b}$$

$$\tau'(1-\omega'g')=\tau(1-\omega g). \tag{17c}$$

We have thus obtained van de Hulst's approximate similarity relations in a physically consistent manner and demonstrated the latters' relationship to a two-term truncation of the phase function for a nonconservative medium.

3. Errors in the delta-Eddington approximation

We define the reflectivity, absorptivity and global transmissivity of an homogeneous layer of optical depth τ_0' to be, respectively,

$$r\equiv F^{\uparrow}(0)/\mu_0\pi F_0, \tag{18a}$$

$$a\equiv[F(0)-F(\tau_0')]/\mu_0\pi F_0, \tag{18b}$$

$$t\equiv F_t^{\downarrow}(\tau_0')/\mu_0\pi F_0, \tag{18c}$$

where $F^{\uparrow}$ is upward flux; $F_t^{\downarrow}$ is global downward flux, consisting of a diffuse $F^{\downarrow}$ and a "direct" component, i.e.,

$$F_t^{\downarrow}(\tau')=F^{\downarrow}(\tau')+\mu_0\pi F_0\exp(-\tau'/\mu_0), \tag{19}$$

and $F=F_t^{\downarrow}-F^{\uparrow}$ is net flux. Note that the "direct" flux in Eq. (19), employing the scaled optical depth $\tau'\leqslant\tau$ [Eq. (13)], is larger than the actual direct flux. On account of the phase function truncation, this "direct" flux also includes scattered radiation traveling in very nearly the same direction as the incident beam. For example, in a dusty atmosphere, a substantial fraction of the solar aureole would be included in the delta-Eddington direct flux. This might actually be an advantage in applying delta-Eddington to the analysis of atmospheric radiation measurements, since it would remove the burden of carefully discriminating against the aureole in measuring direct flux. [In any case, because of the finite angular width of the sun ($\frac{1}{2}°$), it is impossible to entirely distinguish almost-forward-scattered radiation from direct radiation.]

The quantities r, a and t, on account of their definitions (18a)–(18c), satisfy the relationship

$$r+(1-A)t+a=1, \tag{20}$$

where A is the Lambertian albedo of the surface. Since Eq. (20) must hold for exact *and* for approximate solutions, then

$$\Delta r+(1-A)\Delta t+\Delta a=0, \tag{21}$$

where Δ denotes the difference between exact and approximate values. Thus Δr, Δt and Δa may not all have the same sign, and the various errors must compensate one another. This may be observed in all the comparison plots in Figs. 1–3.

The exact values of r, a and t (plotted as solid curves in Figs. 1, 2 and 3) are obtained from the doubling method using 16 Gaussian angles, the diamond initialization and Grant's renormalization method (cf. Wiscombe, 1976); the doubling results have an accuracy of better than 0.1%. These doubling values are compared with conventional Eddington and delta-Eddington values of the same quantities. (The conventional

Eddington approximation is sometimes used for highly asymmetric scattering and an indication of its usefulness is thus called for.) The parameter ranges for which these comparisons were made are shown in Table 1; they include the significant values one will encounter in a planetary atmosphere.

Fig. 1 shows r, a and t as functions of μ_0 for an asymmetry factor $g=0.8$ and a surface albedo $A=0$. Fig. 2 shows similar information for $g=0.95$. (Clouds and aerosols have asymmetry factors between 0.8 and 0.95 for most of the solar spectrum.) Since 90% of the phase function is being truncated in the $g=0.95$ case, because $f=g^2$ we felt it important to also show results for the less drastic $g=0.8$ case. Only $A=0$ is considered because the change in the delta-Eddington error with A is slight; increasing the albedo all the way to $A=0.8$ worsens that error for t and a by no more than 0.01–0.02 and actually decreases the error in r for small and intermediate optical depths (where the surface albedo strongly affects the system albedo). The optical depth τ_0 increases from 0.1 to 1 to 10 across the rows of Figs.

TABLE 1. Parameter ranges over which the delta-Eddington approximation was validated.

g	0 – 0.95
τ_0	0.01–100
ω	0.1 – 0.99
μ_0	0.1 – 1.0
A	0 – 0.8

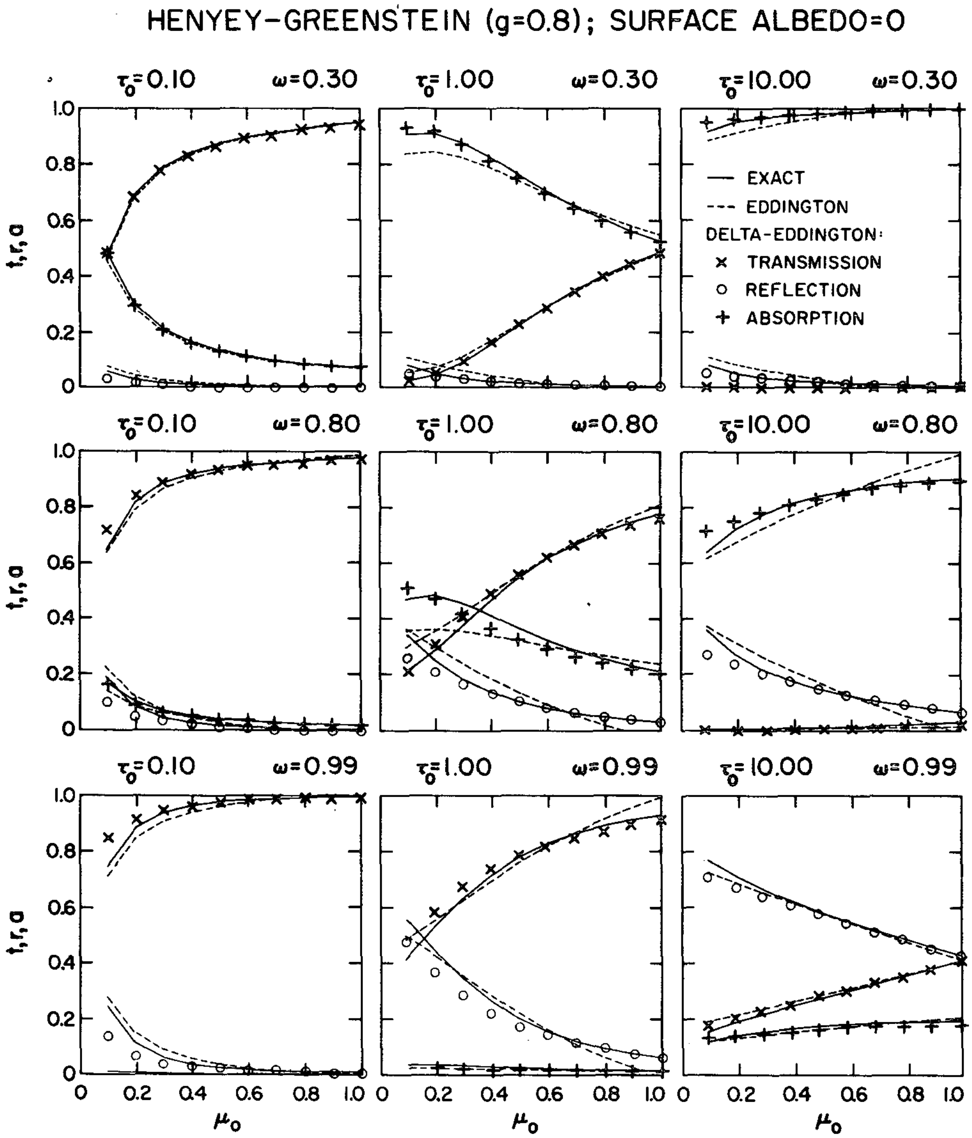

FIG. 1. Reflectivity (r), transmissivity (t) and absorptivity (a) as a function of sun angle μ_0 for various single-scattering albedos (ω) and layer optical depths (τ_0), comparing exact, Eddington and delta-Eddington methods for asymmetry factor $g=0.8$ and surface albedo $A=0$.

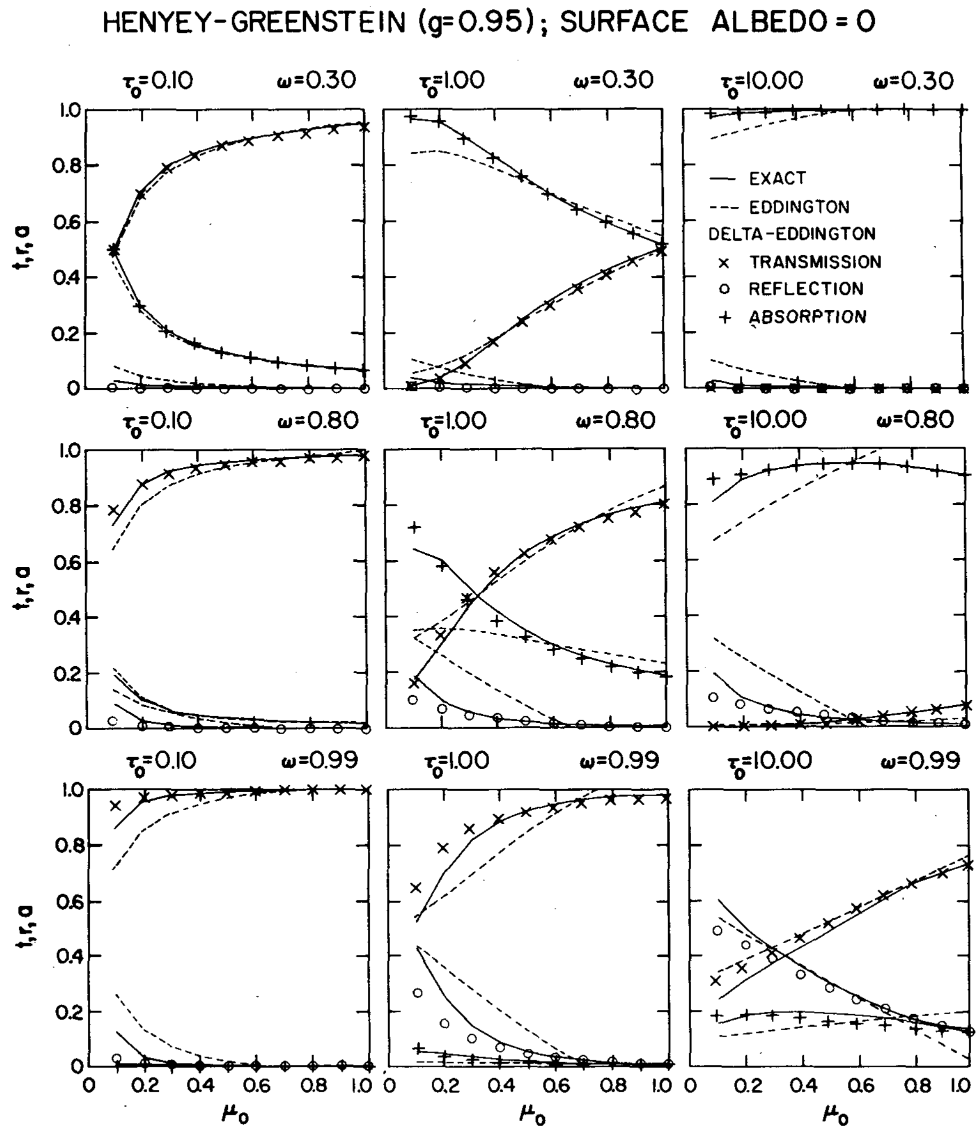

FIG. 2. As in Fig. 1 except for $g=0.95$.

1 and 2, and the single-scattering albedo ω increases from 0.3 to 0.8 to 0.99 down the columns. These values provide a representative sampling of the ω and τ_0 parameter ranges. Curves for $\omega=0.1$ are omitted because in that case the Eddington, delta-Eddington and exact results are virtually indistinguishable except for small μ_0, where the delta-Eddington approximation is substantially better than the Eddington.

Figs. 1 and 2 show that the conventional Eddington approximation is unacceptable for optical depths <10. Eddington predictions of r in a majority of the cases become negative as $\mu_0 \rightarrow 1$, and in some cases values of t or a are predicted to be greater than 1. Comparison of corresponding plots in Figs. 1 and 2 furthermore shows the rapid deterioration of the Eddington approximation with increasing g. By contrast, the delta-Eddington approximation follows the exact curves very well except in the grazing-incidence limit $\mu_0 \rightarrow 0.1$. It should be remembered, however, that the actual radiation fluxes are the product of $\mu_0 \pi F_0$ with r, a and t [cf. Eqs. (18)] so that the delta-Eddington relative flux errors are scaled down by a factor μ_0 compared to the errors shown in Figs. 1 and 2. In our extensive intercomparisons, only a small sample of which are shown here, we have never observed a delta-Eddington flux error exceeding 2.5% of the incident flux πF_0.

Increasing error as $\mu_0 \rightarrow 0$ is typical of both the Eddington approximation (see Wiscombe *et al.*, 1976) and of truncated peak approximations (see, Hansen, 1969; Potter, 1970). Since these two approximations are employed in the delta-Eddington, it shares their deficiency. The reflected radiation in the case of near-grazing incident radiation normally contains a large component of single scattering into the forward peak, which is missed entirely when the forward peak is truncated. Furthermore, this singly-scattered reflected radiation corresponds least to the near isotropy assumed in the Eddington approximation. Because of these facts, the delta-Eddington prediction of r tends to be considerably more in error as $\mu_0 \to 0$ than its predictions of t and a for moderate surface albedos and for single-scattering albedos differing from unity.

The pattern of delta-Eddington error in Figs. 1 and 2 is different for t and r as compared to a. As ω increases (down the columns) t and r retain an accuracy of 0.02 or better for $\mu_0 \geqslant 0.5$, but for $\mu_0 < 0.5$ they deviate increasingly from the exact results. On the other hand, the absorptivity a retains an accuracy of 0.01 or better over almost the full range of μ_0; only for $\mu_0 < 0.2$ or for the mid-ranges of τ_0 and ω do we see errors in a occasionally becoming as large as 0.08 (for $\mu_0 = 0.1$) or 0.04 (all other cases). This renders the computation of heating rates relatively reliable.

When $\mu_0 < 0.5$, the delta-Eddington reflectivity is systematically too low and the delta-Eddington transmissivity almost always systematically too high. The delta-Eddington absorption is systematically too high for $\mu_0 < 0.2$ and systematically too low for $\mu_0 > 0.3$.

In order to examine the problem of small μ_0 from a different viewpoint, we have plotted the doubling and delta-Eddington r, a and t versus optical depth τ_0 in Fig. 3, for an asymmetry factor $g = 0.85$. A surface albedo $A = 0.8$ is used in order to show an opposite extreme from Figs. 1 and 2, where $A = 0$. The sun zenith angle cosine μ_0 increases from 0.1 to 0.4 down the columns of Fig. 3, while the single-scattering albedo ω

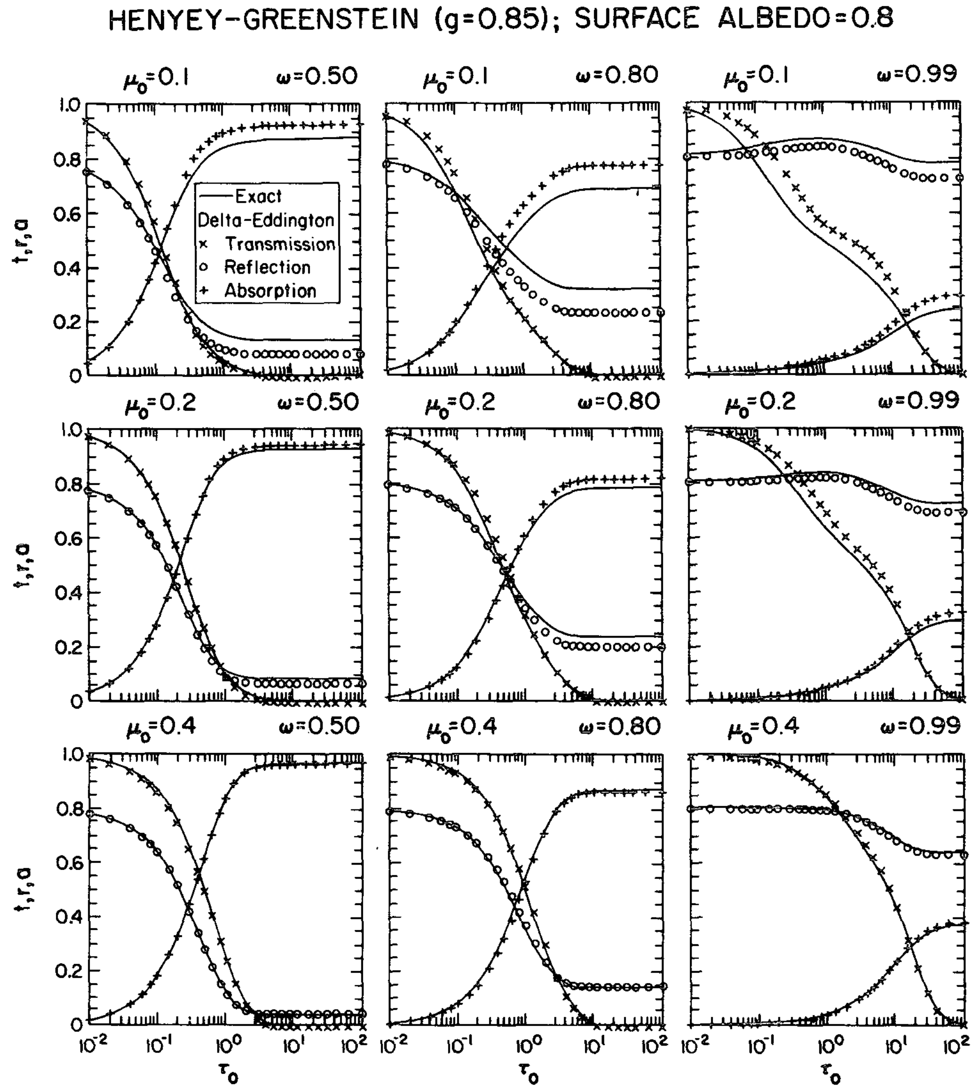

FIG. 3. Reflectivity (r), transmissivity (t) and absorptivity (a) versus layer optical depth τ_0 for a range of large solar zenith angles (μ_0) and larger single-scattering albedos (ω), comparing exact and delta-Eddington methods for asymmetry factor $g = 0.85$ and surface albedo $A = 0.8$.

increases from 0.5 to 0.8 to 0.99 across the rows. As we look down the columns of Fig. 3, we note the dramatic reduction in the delta-Eddington errors as μ_0 increases, for all values of optical depth. By the bottom row ($\mu_0 = 0.4$) the errors are at the 0–0.02 level, and they remain at this level from $\mu_0 = 0.4$ all the way to $\mu_0 = 1$. As ω decreases across the rows, the delta-Eddington errors decrease also; this improvement in accuracy is the more marked the smaller μ_0 becomes, while for $\mu_0 \geqslant 0.4$ it is barely perceptible to plotting accuracy. These trends continue all the way to $\omega = 0$.

While the maximum errors for the large-surface albedo case of Fig. 3 are no greater than for the zero-surface albedo cases of Figs. 1 and 2, there has been a shift in the distribution of error. For $A = 0$ the reflection r exhibited considerably larger errors than t or a, but for $A = 0.8$ this is no longer true—the transmission, and especially the absorption, tend to be every bit as much in error as the reflection.

4. Average error

If the delta-Eddington approximation were applied to a planetary atmosphere, it would be exercised for wide ranges of optical depth and single-scattering albedo. Upon integration over wavelength, over a diurnal course, over a geographically extended area, or any combination of these, individual delta-Eddington errors will average out to some net error. In this section, we apply an averaging procedure which crudely mimics this effect.

Let us define the average relative reflected-flux error as

$$\frac{\langle F^{\uparrow}_{\text{doubling}}(0) - F^{\uparrow}_{\delta\text{-Edd}}(0)\rangle}{\pi F_0} \equiv \frac{1}{N_{\tau_0} N_{\omega} N_{\mu_0}} \sum_{\tau_0} \sum_{\omega} \sum_{\mu_0} \mu_0 \left| r_{\text{doubling}} - r_{\delta\text{-Edd}} \right|.$$

Similar definitions obtain for the average relative transmitted- and absorbed-flux error. For the set of μ_0's, we chose successively increasing ranges $\{0.1\}$, $\{0.1 \rightarrow 0.3\}$, $\{0.1 \rightarrow 0.5\}$, $\{0.1 \rightarrow 0.7\}$ and $\{0.1 \rightarrow 1\}$ to simulate the increasing range of sun angle one encounters in going from pole to equator. We averaged over $N_{\omega} = 11$ values of ω uniformly distributed between 0 and 1, and over $N_{\tau_0} = 32$ values of τ_0. Nine values of τ_0 were taken in each of the decades [0.1,1], [1,10] and [10,100], while only five values were taken in the decade [0.01,0.1] because of its relative unimportance.

Selected but representative average errors are shown in Table 2 (some of the variations in this table seem erratic because all entries have been rounded to two digits). These average flux errors range between 0.1% and 0.5% of the incident flux. They do not exhibit the serious loss of accuracy for small solar elevations displayed by the flux ratios r, a and t, which bears out our earlier comment about flux errors being scaled down by a factor μ_0; and they generally decrease as g increases, which bears out our discussion in Section 2 [see Eq. (7)]. The average reflected and absorbed flux errors are very nearly equal and usually exceed the average transmitted flux error. In general, for $A = 0$ all the errors are fairly insensitive to the increasing range of sun angle (barring the $\mu_0 = 0.1$ case), but when $A = 0.8$ all errors increase significantly with increasing range of μ_0. As A increases, errors for $\mu_0 \rightarrow 0$ are generally diminished, while those for $\mu_0 \rightarrow 1$ are enhanced.

TABLE 2. Average percent error in reflected (ref), transmitted (trn) and absorbed (abs) delta-Eddington fluxes for surface albedos $A = 0$ and $A = 0.8$, asymmetry factors $g = 0.8$ and $g = 0.95$, and various ranges of solar zenith angle cosine μ_0.

	$g=0.8$			$g=0.95$		
μ_0	ref	trn	abs	ref	trn	abs
			$A=0$			
0.1	0.42	0.13	0.30	0.40	0.15	0.28
0.1–0.3	0.31	0.20	0.23	0.25	0.19	0.20
0.1–0.5	0.27	0.22	0.28	0.21	0.18	0.22
0.1–0.7	0.29	0.21	0.33	0.20	0.16	0.23
0.1–1.0	0.29	0.17	0.32	0.18	0.13	0.21
			$A=0.8$			
0.1	0.33	0.12	0.31	0.31	0.14	0.29
0.1–0.3	0.22	0.16	0.21	0.18	0.16	0.18
0.1–0.5	0.22	0.18	0.23	0.18	0.18	0.19
0.1–0.7	0.28	0.25	0.28	0.22	0.26	0.24
0.1–1.0	0.33	0.41	0.35	0.28	0.42	0.33

5. Summary and conclusions

We have presented an extension of the Eddington approximation in which the forward peak of the phase function is approximated as a δ-function. The approximate phase function is constructed to have the same asymmetry factor and second moment as the actual phase function. For Henyey–Greenstein phase functions this leads to a scaling of the asymmetry factor, optical depth and single-scattering albedo which can be related to some recent "similarity relations" of van de Hulst. In particular, the transformed asymmetry factor assumes a value between 0 and 0.5 for which we demonstrated that the conventional Eddington approximation is good (see Wiscombe *et al.*, 1976).

The transmission, reflection and absorption predicted by our approximation were compared with doubling method calculations for realistic ranges of optical depth, single-scattering albedo, surface albedo, sun angle and asymmetry factor. In all cases, all relevant fluxes are predicted to an accuracy of better than 2.5% of the incident flux with the "average" flux error (see Section 4) being no larger than 0.5%. The flux ratios—global transmissivity, reflectivity and absorptivity—are accurate to at least 0.02 when the solar zenith angle cosine $\mu_0 \geqslant 0.4$, but these quantities experience errors increasing to a maximum of 0.15 (in the reflectivity) as $\mu_0 \rightarrow 0.1$. This loss of accuracy practically disappears, however, for fluxes rather than flux ratios. Also, our approximation never exhibits negative reflectivities, or transmissivities and absorptivities exceeding 100%, which may occur in the conventional Eddington approximation for large asymmetry factors.

The delta-Eddington approximation therefore furnishes a physically sound, accurate and analytically simple parameterization of radiation to replace the empiricism currently employed in many general circulation and climate models. While attention has been confined in this paper to homogeneous layers, vertical inhomogeneity may be simply treated by concatenating homogeneous layers and imposing flux continuity at the layer boundaries, in the manner of Shettle and Weinman (1970). Well-documented computer codes are available from the authors for monochromatic n-layer cases.

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