Lecture 2.

The Beer-Bouguer-Lambert law.

Concepts of extinction (scattering plus absorption) and emission.

Schwarzschild's equation.

Objectives:

1. The Beer-Bouguer-Lambert law. Concepts of extinction (scattering + absorption) and emission. Optical depth.
2. A differential form of the radiative transfer equation (Schwarzschild's radiative transfer equation).

Required reading:

L02: 1.1, 1.4

1. The Beer-Bouguer-Lambert law. Concepts of extinction (scattering + absorption) and emission.

- **Extinction** and **emission** are two main types of the interaction between an electromagnetic radiation field and a medium (e.g., the atmosphere).

General definition:

Extinction is a process that decreases the radiant **intensity**, while **emission** increases it.

NOTE: "same name": **extinction = attenuation**

Radiation is **emitted** by **all** bodies that have a temperature above absolute zero (O K) (called **thermal emission**).

- **Extinction** is due to **absorption** and **scattering**.

Absorption is a process that removes the radiant energy from an electromagnetic field and transfers it to other forms of energy.

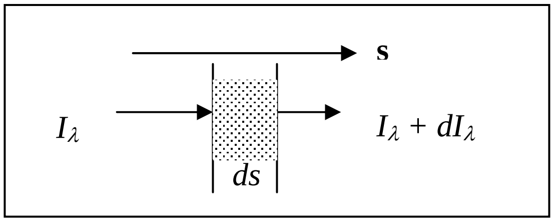
Scattering is a process that **does not** remove energy from the radiation field, but may redirect it.

NOTE: Scattering can be thought of as **absorption** of radiant energy followed by **re-emission** back to the electromagnetic field with negligible conversion of energy. Thus, scattering can remove radiant energy of a light beam traveling in one direction, but can be a "source" of radiant energy for the light beams traveling in other directions.

The fundamental law of extinction is the **Beer-Bouguer-Lambert (Extinction) law,** which states that the extinction process is linear in the intensity of radiation and amount of radiatively active matter, provided that the physical state (i.e., T, P, composition) is held constant.

NOTE: Some non-linear processes do occur as will be discussed later in the course.

Consider a small volume ΔV of infinitesimal length ds and area ΔA containing radiatively active matter. The change of intensity along the path ds is proportional to the amount of matter in the path.



For extinction: $$dI_\lambda = -\beta_{e,\lambda} I_\lambda ds \quad [2.1]$$

For emission: $$dI_\lambda = \beta_{e,\lambda} J_\lambda ds \quad [2.2]$$

where $\beta_{e,\lambda}$ is the **volume extinction coefficient** (LENGTH^{-1}) and J_λ is the **source function**.

- The **source function** J_λ has emission and scattering contributions or only scattering.

- Generally, the **volume extinction coefficient** is a function of position **s.**

NOTE: Volume extinction coefficient is often referred to as the **extinction coefficient.**

Extinction coefficient = absorption coefficient + scattering coefficient

$$\beta_{e,\lambda} = \beta_{a,\lambda} + \beta_{s,\lambda} \qquad [2.3]$$

NOTE: Extinction coefficient (as well as absorption and scattering coefficients) can be expressed in different forms according to the definition of the amount of matter (e.g., number concentrations, mass concentration, etc.) of matter in the path.

- **Volume** and **mass extinction coefficients** are most often used.

Mass extinction coefficient = volume extinction coefficient/density

UNITS: the mass coefficient is in unit area per unit mass ($\mathrm{LENGTH}^2\ \mathrm{MASS}^{-1}$). For instance: ($cm^2\ g^{-1}$), ($m^2\ kg^{-1}$), etc.

If ρ is the density (mass concentration) of a given type of particles (or molecules), then

$$\beta_{e,\lambda} = \rho k_{e,\lambda}$$
$$\beta_{s,\lambda} = \rho k_{s,\lambda} \qquad [2.4]$$
$$\beta_{a,\lambda} = \rho k_{a,\lambda}$$

where the $k_{e,\lambda}$; $k_{s,\lambda}$, and $k_{a,\lambda}$ are the **mass extinction, scattering, and absorption coefficients**, respectively.

NOTE: L02 uses k_λ for both mass extinction and mass absorption coefficients!

Using the mass extinction coefficient, the **Beer-Bouguer-Lambert (extinction) law (Eqs.[2.1]-2.2])** is

$$dI_\lambda = -\rho k_{e,\lambda} I_\lambda ds \qquad [2.5]$$
$$dI_\lambda = \rho k_{e,\lambda} J_\lambda ds \qquad [2.6]$$

The **extinction cross section** of a given particle (or molecule) is a parameter that measures the attenuation of electromagnetic radiation by this particle (or molecule). In the same fashion, **scattering and absorption cross sections** can be defined.

UNITS: the cross section is in unit area (LENGTH^2)

If N is the particle (or molecule) number concentration of a given type of particles (or molecules) of the same size, then

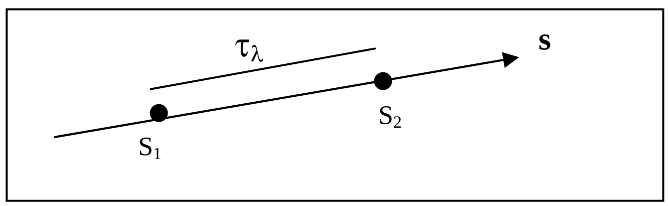
$$\beta_{e,\lambda} = \sigma_{e,\lambda} N$$
$$\beta_{s,\lambda} = \sigma_{s,\lambda} N \qquad [2.7]$$
$$\beta_{a,\lambda} = \sigma_{a,\lambda} N$$

where $\sigma_{e,\lambda}$, $\sigma_{s,\lambda}$, and $\sigma_{a,\lambda}$ are the extinction, scattering, and absorbing cross sections, respectively.

UNITS: Particle (molecule) number concentration, N, is in the number of particles (molecules) per unit volume (LENGTH^{-3}).

Optical depth of a medium between points s_1 and s_2 is defined as

$$\tau_\lambda(s_2; s_1) = \int_{s_1}^{s_2} \beta_{e,\lambda}(s)\,ds$$



UNITS: optical depth is unitless.

NOTE: "same name": **optical depth = optical thickness = optical path**

If $\beta_{e,\lambda}$ (s) does not depend on position (called a homogeneous optical path), thus

$\beta_{e,\lambda}(s) = \beta_{e,\lambda}$ and $\tau_\lambda(s_2; s_1) = \beta_{e,\lambda}(s_2 - s_1) = \beta_{e,\lambda} s$

In this case by integrating Eq.[2.1], the **Extinction law** can be expressed as

$$I_\lambda = I_0 \exp(-\tau) = I_0 \exp(-\beta_{e,\lambda} s) \qquad [2.8]$$

Optical depth can be expressed in several ways:

$$\tau_\lambda(s_1;s_2) = \int_{s_1}^{s_2} \beta_{e,\lambda} ds = \int_{s_1}^{s_2} \rho k_{e,\lambda} ds = \int_{s_1}^{s_2} N\sigma_{e,\lambda} ds \qquad [2.9]$$

- If in a given volume there are several types of optically active particles each with $\boldsymbol{\beta^{i}_{e,\lambda}}$**,** etc., then the optical depth can be expressed as:

$$\tau_\lambda = \sum_i \int_{s_1}^{s_2} \beta^i_{e,\lambda}\ ds = \sum_i \int_{s_1}^{s_2} \rho_i k^i_{e,\lambda} ds = \sum_i \int_{s_1}^{s_2} N_i \sigma^i_{e,\lambda} ds \qquad [2.10]$$

where $\boldsymbol{\rho_i}$ and $\boldsymbol{N_i}$ is the mass concentrations (densities) and particles concentrations of the ***i***-th species.

2. A differential form of the radiative transfer equation

Consider a small volume ΔV of infinitesimal length ***ds*** and area ΔA containing radiatively active matter. Using the **Extinction law**, the change (loss plus gain due to both the thermal emission and scattering) of intensity along the path ***ds*** is

$$dI_\lambda = -\beta_{e,\lambda} I_\lambda ds + \beta_{e,\lambda} J_\lambda ds$$

Dividing by $\boldsymbol{\beta_{e,\lambda} ds}$, we find

$$\boxed{\frac{dI_\lambda}{\beta_{e,\lambda} ds} = -I_\lambda + J_\lambda} \qquad [2.11]$$

Eq. [2.11] is the **differential equation of radiative transfer (called Schwarzchild's equation).**

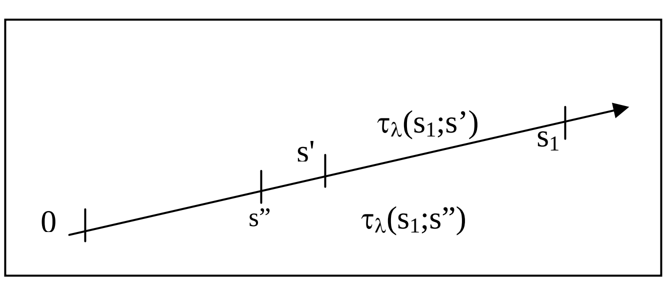
NOTE: Both $\boldsymbol{I_\lambda}$ and $\boldsymbol{J_\lambda}$ are generally functions of both position and direction.

The optical depth is

$$\tau_\lambda(s_1;s) = \int_{s}^{s_1} \beta_{e,\lambda}(s) ds$$

Thus

$$d\tau_\lambda = -\beta_{e,\lambda}(s) ds$$



Using the above expression for $d\tau_\lambda$, we can re-write Eq. [2.11] as

$$-\frac{dI_\lambda}{d\tau_\lambda} = -I_\lambda + J_\lambda$$

or as [2.12]

$$\frac{dI_\lambda}{d\tau_\lambda} = I_\lambda - J_\lambda$$

These are other forms of the **differential equation of radiative transfer.**

Let's re-arrange terms in the above equation and multiply both sides by ***exp(-τ_λ)***. We have

$$-\frac{\exp(-\tau_\lambda)dI_\lambda}{d\tau_\lambda} + \exp(-\tau_\lambda)I_\lambda = \exp(-\tau_\lambda)J_\lambda$$

and (using that ***d[I(x)exp(-x)]=exp(-x)dI(x)-exp(-x)I(x)dx***) we find

$$-d\left[I_\lambda \exp(-\tau_\lambda)\right] = \exp(-\tau_\lambda)J_\lambda d\tau_\lambda$$

Then integrating over the path from **0** to **s_1 ,** we have

$$-\int_0^{s_1} d\left[I_\lambda(s)\exp(-\tau_\lambda(s_1;s))\right] = \int_0^{s_1}\exp(-\tau_\lambda(s_1;s))J_\lambda d\tau_\lambda$$

and

$$-\left[I_\lambda(s_1) - I_\lambda(0)\exp(-\tau_\lambda(s_1;0))\right] = \int_0^{s_1}\exp(-\tau_\lambda(s_1;s))J_\lambda d\tau_\lambda$$

Thus

$$I_\lambda(s_1) = I_\lambda(0)\exp(-\tau_\lambda(s_1;0)) - \int_0^{s_1}\exp(-\tau_\lambda(s_1;s))J_\lambda d\tau_\lambda$$

and, using $d\tau_\lambda = -\beta_{e,\lambda}(s)ds$, we have a **solution** of the **equation of radiative transfer (often referred to as the integral form of the radiative transfer equation):**

$$I_\lambda(s_1) = I_\lambda(0)\exp(-\tau_\lambda(s_1;0)) + \int_0^{s_1}\exp(-\tau_\lambda(s_1;s))J_\lambda \beta_{e,\lambda} ds \qquad [2.13]$$

NOTE:

i) The above equation gives monochromatic intensity at a given point propagating in a given direction (often called an **elementary solution**). A completely general distribution of intensity in angle and wavelengths (or frequencies) can be obtained by repeating the elementary solution for all incident beams and for all wavelengths (or frequencies).

ii) Knowledge of the **source function** J_λ is required to solve the above equation. In the general case, the source function consists of thermal emission and scattering (or from scattering), depends on the position and direction, and is very complex. One may say that the radiative transfer equation is all about the source function.

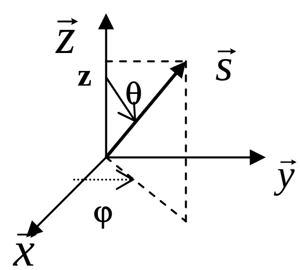
- **Plane-parallel atmosphere.**

✓ For many applications, the atmosphere can be approximated by **a plane-parallel model** to handle the vertical stratification of the atmosphere.

The plane-parallel atmosphere consists of a certain number of atmospheric layers each characterized by homogeneous properties (e.g., T, P, optical properties of a given species, etc.) and bordered by the bottom and top infinite plates (called boundaries).

- Traditionally, the **vertical coordinate** ***z*** is used to measure linear distances in the plane-parallel atmosphere:

$$z = s\cos(\theta)$$



where θ denotes the angle between the upward normal and the direction of propagation of a light beam (or zenith angle) and φ is the azimuthal angle.

Using $ds = dz/\cos(\theta)$, the **radiative transfer equation** can be written as

$$\cos(\theta)\frac{dI_\lambda(z;\theta;\varphi)}{\beta_{e,\lambda}dz} = -I_\lambda(z;\theta;\varphi) + J_\lambda(z;\theta;\varphi)$$

Introducing the optical depth measured from the outer boundary downward as

$$\tau_\lambda(z_1; z) = \int_z^{z_1} \beta_{e,\lambda}(z)dz$$

and using $d\tau_\lambda = -\beta_{e,\lambda}(z)dz$ and $\boldsymbol{\mu = \cos(\theta)}$**,** we have

$$\boxed{\mu \frac{dI_\lambda(\tau;\mu;\varphi)}{d\tau} = I_\lambda(\tau;\mu;\varphi) - J_\lambda(\tau;\mu;\varphi)} \qquad [2.13]$$

- Eq.[2.13] may be solved to give the **upward (or upwelling) and downward (or downwelling) intensities** for a finite atmosphere which is bounded on two sites.

Upward intensity $I_\lambda^{\uparrow}$ is for $1 \geq \mu \geq 0$ (or $0 \leq \theta \leq \pi/2$);

Downward intensity $I_\lambda^{\downarrow}$ is for $-1 \leq \mu \leq 0$ (or $\pi/2 \leq \theta \leq \pi$)

(using that ***cos***(0)=1; ***cos***(***π***/2)=0 and ***cos***(***π***) =-1)

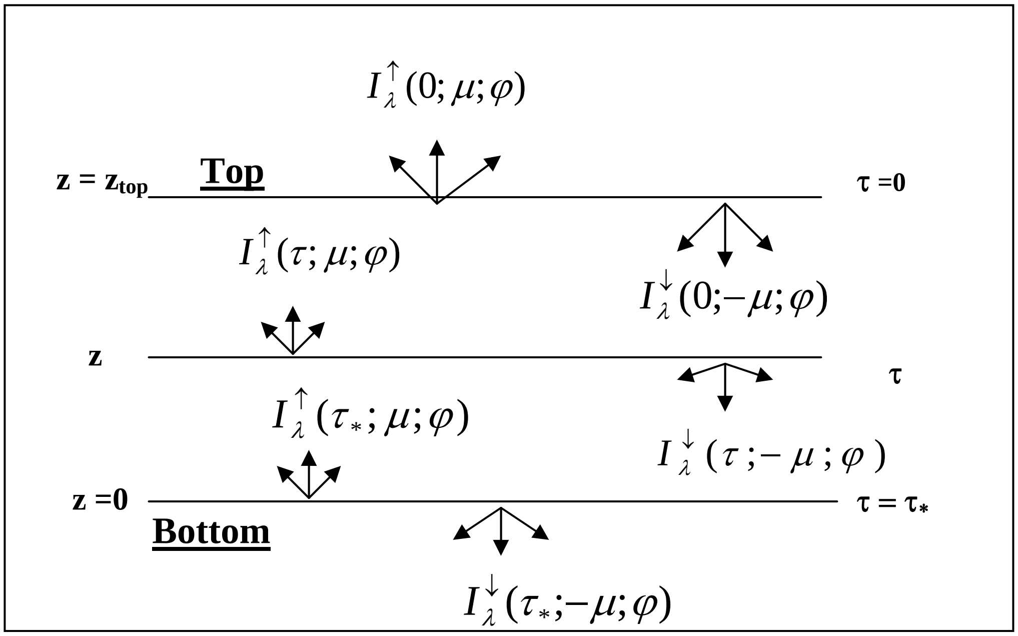


Figure 2.1 Schematic representation of the plane-parallel atmosphere.

NOTE: For downward intensity, μ is replaced by **–μ**.

The **radiative transfer equation** [2.13] can be written for **upward and downward intensities:**

$$\mu \frac{dI_\lambda^{\uparrow}(\tau;\mu;\varphi)}{d\tau} = I_\lambda^{\uparrow}(\tau;\mu;\varphi) - J_\lambda^{\uparrow}(\tau;\mu;\varphi) \quad [2.14a]$$

$$-\mu \frac{dI_\lambda^{\downarrow}(\tau;-\mu;\varphi)}{d\tau} = I_\lambda^{\downarrow}(\tau;-\mu;\varphi) - J_\lambda^{\downarrow}(\tau;-\mu;\varphi) \quad [2.14b]$$

A solution of Eq.[2.14a] gives a upward intensity in the plane-parallel atmosphere:

$$I_\lambda^{\uparrow}(\tau;\mu;\varphi) = I_\lambda^{\uparrow}(\tau_*;\mu;\varphi)\exp(-\frac{\tau_* - \tau}{\mu}) + \frac{1}{\mu}\int_{\tau}^{\tau_*}\exp(-\frac{\tau' - \tau}{\mu})\, J_\lambda^{\uparrow}(\tau';\mu;\varphi)\, d\tau' \quad [2.15a]$$

and **a solution of Eq.[2.14b] gives a downward intensity in the plane-parallel atmosphere:**

$$I_\lambda^{\downarrow}(\tau;-\mu;\varphi) = I_\lambda^{\downarrow}(0;-\mu;\varphi)\exp(-\frac{\tau}{\mu}) + \frac{1}{\mu}\int_{0}^{\tau}\exp(-\frac{\tau - \tau'}{\mu})\, J_\lambda^{\downarrow}(\tau';-\mu;\varphi)\, d\tau' \quad [2.15b]$$