Lecture 21.

Methods for solving the radiative transfer equation with multiple scattering. Part 3: "Exact" methods: Discrete-ordinate and Adding.

Objectives:

1. Discrete-ordinate method for the case of isotropic scattering.
2. Generalization of the discrete-ordinate method for inhomogeneous atmosphere.
3. Numerical implementation of the discrete-ordinate method: DISORT
4. Principles of invariance.
5. Adding method.

Required reading:

L02: 6.2, 6.3.1- 6.3.4, 6.4

Ricchazzi, P., S. R. Yang, et al. SBDART: A research and teaching software tool for Plane-parallell radiative transfer in the earth's atmosphere. Bulletin of the American Meteorological Society 79, 2101-2114, 1998.

http://www.icess.ucsb.edu/esrg/SBDART.html

Advanced reading:

Thomas G.E. and K. Stamnes, Radiative transfer in the atmosphere and ocean, 2000, Chapter 8.1-8.10 .

Liu Q. and F. Weng, Advanced Doubling–Adding Method for Radiative Transfer in Planetary Atmospheres. Journal of Atmospheric Sciences 63, 3459–3465, 2006.

Stamnes, K., S. Tsay, W. Wiscombe and K. Jayaweera, Numerically stable algorithm for discrete-ordinate-method radiative transfer in multiple scattering and emitting layered media." Appl. Opt., 27, 2502-2509, 1998.

1. Discrete-ordinate method for the case of isotropic scattering.

NOTE: A discrete-ordinate method has been developed by Chandrasekhar in the 1950s (Chandrasekhar S., Radiative transfer, 1960, Dover Publications).

Recall the radiative transfer equation (Lecture 18) for azimuthally independent diffuse intensity:

$$\mu \frac{dI(\tau,\mu)}{d\tau} = I(\tau,\mu) - \frac{\omega_0}{2}\int_{-1}^{1} I(\tau,\mu')P(\mu,\mu')d\mu' - \frac{\omega_0}{4\pi}F_0 P(\mu,-\mu_0)\exp(-\tau/\mu_0)$$

For isotropic scattering, the scattering phase function is 1. Hence we have

$$\mu \frac{dI(\tau,\mu)}{d\tau} = I(\tau,\mu) - \frac{\omega_0}{2}\int_{-1}^{1} I(\tau,\mu')d\mu' - \frac{\omega_0 F_0}{4\pi}\exp(-\tau/\mu_0) \qquad [21.1]$$

Let's apply the Gaussian quadratures to replace the integral in Eq.[21.1]

$$\mu_i \frac{dI(\tau,\mu_i)}{d\tau} = I(\tau,\mu_i) - \frac{\omega_0}{2}\sum_{j=-n}^{n} a_j I(\tau,\mu_j) - \frac{\omega_0 F_0}{4\pi}\exp(-\tau/\mu_0) \qquad [21.2]$$

Inhomogeneous part

where i=-n, …,n (2n terms) and $\boldsymbol{a_j}$ are the Gaussian weights (constants) and $\boldsymbol{\mu_j}$ are quadrature angles (or points).

Eq.[21.2] is a system of 2n inhomogeneous differential equations:

Solution of Eq.[21.2] = general solution + particular solution

where the general solution is a solution of the homogeneous part of the Eq.[21.2]

Denoting $\boldsymbol{I_i = I_i(\tau,\mu_i)}$, the general solution of Eq.[21.2] can be found as

$$I_i = g_i \exp(-k\tau) \qquad [21.3]$$

Inserting Eq.[21.3] into Eq.[21.2], we obtain

$$g_i(1+\mu_i k) = \frac{\omega_0}{2}\sum_{j=-n}^{n} a_j g_j \qquad [21.4]$$

We can find $\boldsymbol{g_i}$ in the form

$$g_i = L/(1+\mu_i k)$$

where L is a constant to be determined. Substituting this expression for $\boldsymbol{g_i}$ in Eq.[21.4], we have

$$1 = \frac{\omega_0}{2}\sum_{j=-n}^{n} \frac{a_j}{1+\mu_i k} = \omega_0 \sum_{j=1}^{n} \frac{a_j}{1-\mu_j^2 k^2} \qquad [21.5]$$

Eq.[21.5] gives 2n solutions for +/-k_j (j=1,…,n).

Thus general solution is

$$I_i = \sum_j \frac{L_j}{1+\mu_i k_j}\exp(-k_j\tau) \qquad [21.6]$$

where $\boldsymbol{L_j}$ are constants.

The particular solution can be found as

$$I_i = \frac{\omega_0 F_0}{4\pi} h_i \exp(-\tau / \mu_0) \qquad [21.7]$$

where ***h_i*** are constants.

Inserting Eq.[21.7] into Eq.[21.2], we have

$$h_i (1 + \mu_i / \mu_0) = \frac{\omega_0}{2} \sum_{j=-n}^{n} a_j h_j + 1 \qquad [21.8]$$

From Eq.[21.8], ***h_i*** is found as

$$h_i = \gamma / (1 + \mu_i / \mu_0)$$

where γ is determined from

$$\gamma = 1 / \{1 - \frac{\omega_0}{2} \sum_{j=1}^{n} a_j / (1 - \mu_j^2 / \mu_0^2)\} \qquad [21.9]$$

Adding the general solution Eq.[21.6] and the particular solution Eq.[21.7], we have

$$I_i = \sum_j \frac{L_j}{1 + \mu_i k_j} \exp(-k_j \tau) + \frac{\omega_0 F_0 \gamma}{4\pi (1 + \mu_i / \mu_0)} \exp(-\tau / \mu_0) \qquad [21.10]$$

where ***L_j*** are constants to be determined from the boundary conditions.

H-function has been introduced by Chandrasekhar as

$$H(\mu) = \frac{1}{\mu_1 \ldots \mu_n} \frac{\prod_{j=1}^{n} (\mu + \mu_j)}{\prod_{j=1}^{1} (1 + k_j \mu)} \qquad [21.11]$$

Expressing γ in the H-function, Eq.[21.10] becomes

$$I_i = \sum_j \frac{L_j}{1 + \mu_i k_j} \exp(-k_j \tau) + \frac{\omega_0 F_0 H(\mu_0) H(-\mu_0)}{4\pi (1 + \mu_i / \mu_0)} \exp(-\tau / \mu_0) \qquad [21.12]$$

Eq.[21.12] gives a simple solution for the semi-infinite isotropic atmosphere (see L02:6.2.2)

$$I^{\uparrow}(0, \mu) = \frac{1}{4\pi} \omega_0 F_0 \frac{\mu_0}{\mu + \mu_0} H(\mu_0) H(\mu) \qquad [21.13]$$

2. Generalization of the discrete-ordinate method for an inhomogeneous atmosphere.

Let's consider the atmosphere with non-isotropic scattering.

We can expand the diffuse intensity in the cosine series

$$I(\tau,\mu,\varphi)=\sum_{m=0}^{N} I^{m}(\tau,\mu)\cos(m(\varphi_0-\varphi))$$

So we need to solve

$$\mu\frac{dI^{m}(\tau,\mu)}{d\tau}=I^{m}(\tau,\mu)-(1+\delta_{0,m})\frac{\omega_0}{4}\sum_{l=m}^{N}\varpi_l^{m}P_l^{m}(\mu)\int_{-1}^{1}P_l^{m}(\mu')I^{m}(\tau,\mu')d\mu'-$$

$$-\frac{\omega_0}{4\pi}\sum_{l=m}^{N}\varpi_l^{m}P_l^{m}(\mu)P_l^{m}(-\mu_o)F_0\exp(-\tau/\mu_0)$$

The **general solution** may be written as

$$I^{m}(\tau,\mu_i)=\sum_{j=-n}^{n}L_j^{m}\phi_j^{m}(\mu_j)\exp(-k_j^{m}\tau)$$

ϕ_j^{m}, k_j^{m}, L_j^{m} are coefficients to be determined.

The **particular solution** may be written as

$$I_p^{m}(\tau,\mu_i)=Z^{m}(\mu_i)\exp(-\tau/\mu_0)$$

Where $Z^{m}(\mu_i)$ is the following function

$$Z^{m}(\mu_i)=\frac{1}{4\pi}\omega_0F_0P_m^{m}(-\mu_0)\frac{H^{m}(\mu_0)H^{m}(-\mu_0)}{1+\mu_i/\mu_0}\sum_{l=0}^{N}\varpi_l^{m}\zeta_l^{m}\frac{1}{\mu_0}P_l^{m}(\mu_i)$$

The **complete solution** of the radiative transfer is

$$I^{m}(\tau,\mu_i)=\sum_{j=-n}^{n}L_j^{m}\phi_j^{m}(\mu_j)\exp(-k_j^{m}\tau)+Z^{m}(\mu_i)\exp(-\tau/\mu_0) \qquad [21.14]$$

i=-*n*,…,*n*

Let's generalize the **complete solution** Eq.[21.14] of the radiative transfer for the inhomogeneous atmosphere. The atmosphere can be divided into the N homogeneous layers, each is characterized by a single scattering albedo, phase function, and optical depth.

NOTE: If an atmospheric layer has gases, aerosols and/or clouds, one needs to calculate the effective optical properties of this layer.

For ***l***-th layer, we can write the solution using Eq.[21.14]. To simplify notations, let's consider the azimuthal independent case (i.e., ***m=0***), so we have

$$I^l(\tau,\mu_i) = \sum_{j=-n}^{n} L_j^l \phi_j^l(\mu_j)\exp(-k_j^l\tau) + Z^l(\mu_i)\exp(-\tau/\mu_0) \qquad [21.15]$$

Now, we need to match the boundary and continuity conditions between layers.

At the top of the atmosphere (TOA): no downward diffuse intensity

$$I^{l=1}(0,-\mu_i) = 0 \qquad [21.16]$$

At the layer's boundary: upward and downward intensities must be continuous

$$I^l(\tau_l,\mu_i) = I^{l+1}(\tau_l,\mu_i) \qquad [21.17]$$

At the bottom of the atmosphere (assuming the Lamdertian surface):

$$I^{l=N}(\tau_N,\mu_i) = \frac{r_{sur}}{\pi}[F^{\downarrow}(\tau_N) + \mu_0 F_0 \exp(-\tau_N/\mu_0)] \qquad [21.18]$$

Eqs.[21.16]-[21.18] provide necessary equations to find the unknown coefficients.

<u>3. Numerical implementation of the discrete-ordinate method: DISORT</u>

DISORT is a FORTRAN numerical code based on the discrete-ordinate method developed by Stamnes, Wiscombe et al.

DISORT is openly available and has a good user-guide.

<u>*Some features:*</u>

1) DISORT applies to the inhomogeneous nonithothermal plane-parallel atmosphere.

2) A user may set-up any numbers of the plane-parallel layers.

3) Each layer must be characterized by the effective optical depth, single scattering albedo and asymmetry parameter if the Henyey-Greenstein phase function is used.

4) A user may use any phase function by providing the Legendre polynomial expansion coefficients.

5) A user selects a number of streams (keeping in mind that the computation time varies as n^3).

6) A key problem is to obtain a solution for fluxes for strongly forward-peaked scattering.

7) DISORT allows predicting the intensity as a function of the direction and position at any point in the atmosphere (i.e., not only at the boundaries of the layers).

DISORT is incorporated into the SBDART radiative transfer code.

SBDART (Santa Barbara DISORT Atmospheric Radiative Transfer) is a FORTRAN computer code: http://www.icess.ucsb.edu/esrg/SBDART.html
(see Introduction to SBDART)

4. Principles of invariance.

Recall the definitions of reflection and transmission of a layer introduced in Lecture 19.

If the solar flux is incident on a layer of optical depth $\tau*$:

$$R(\mu, \varphi, \mu_0, \varphi_0) = \pi I^{\uparrow}(0, \mu, \varphi) / \mu_0 F_0$$

$$T(\mu, \varphi, \mu_0, \varphi_0) = \pi I^{\downarrow}(\tau*, -\mu, \varphi) / \mu_0 F_0$$

Or in the general case:

$$I_r^{\uparrow}(0, \mu, \varphi) = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 R(\mu, \varphi, \mu', \varphi') I_{inc}(-\mu', \varphi') \mu' d\mu' d\varphi'$$

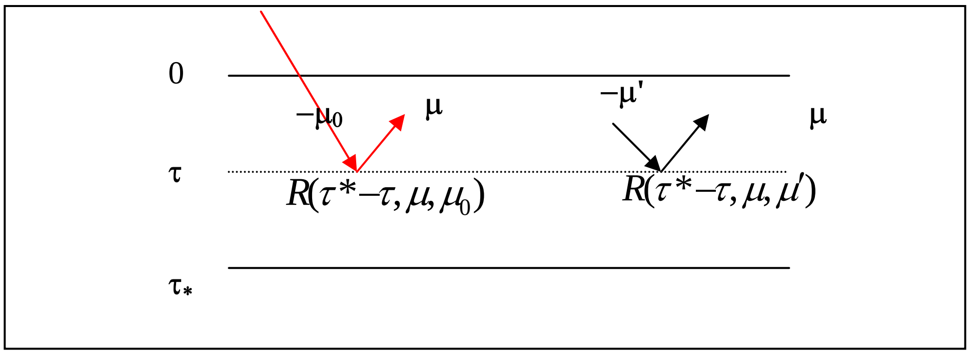
$$I_t^{\downarrow}(\tau*, -\mu, \varphi) = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 T(\mu, \varphi, \mu', \varphi') I_{inc}(-\mu', \varphi') \mu' d\mu' d\varphi'$$

- **The principle of invariance for the <u>semi-infinite</u> atmosphere** (Ambartzumian, 1940): the diffuse reflected intensity cannot be changed if a layer of finite optical depth, having the same optical properties as those of the original layer, is added (see L02: 6.3.2).

➢ ***The principles of invariance for a <u>finite</u> atmosphere*** (Chandrasekhar, 1950):

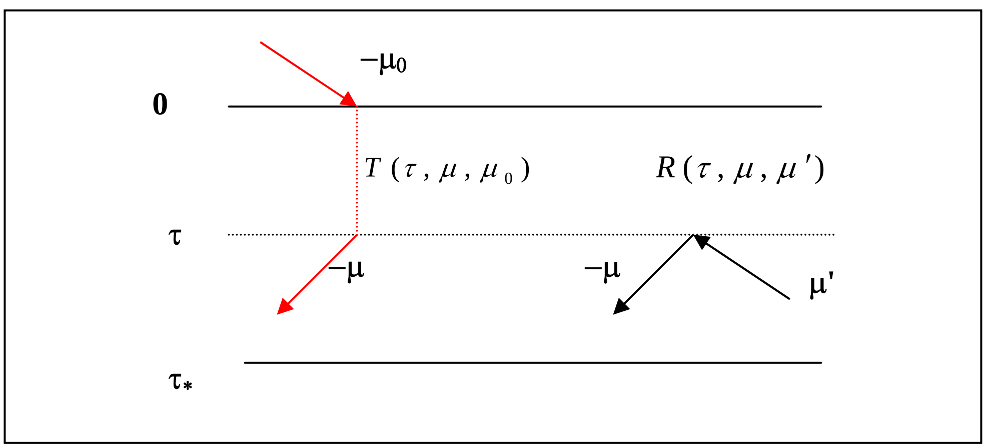
(1) The reflected (upward) intensity at any given optical depth τ results from the reflection of (a) the attenuated solar flux = $\mu_0 F_0 \exp(-\tau/\mu_0)$ and (b) the downward diffuse intensity at the level τ:

$$I^{\uparrow}(\tau,\mu) = \frac{\mu_0 F_0}{\pi}\exp(-\tau/\mu_0) R(\tau_1-\tau,\mu,\mu_0) + 2\int_0^1 R(\tau^*-\tau,\mu,\mu') I^{\downarrow}(\tau,-\mu')\mu' d\mu' \quad \textbf{[21.19]}$$



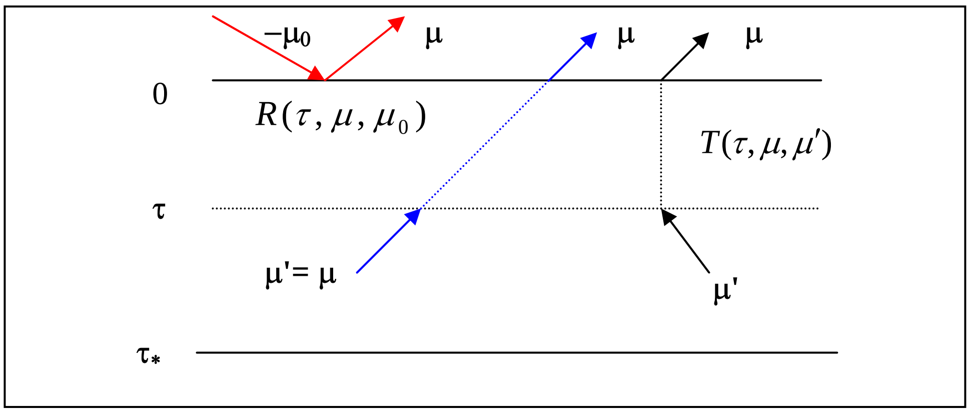
(2) The diffusely transmitted (downward) intensity at the level τ results from (a) the transmission of incident solar flux and (b) the reflection of the upward diffuse intensity above the level τ:

$$I^{\downarrow}(\tau,-\mu) = \frac{\mu_0 F_0}{\pi} T(\tau,\mu,\mu_0) + 2\int_0^1 R(\tau,\mu,\mu') I^{\uparrow}(\tau,\mu')\mu' d\mu' \quad [21.20]$$



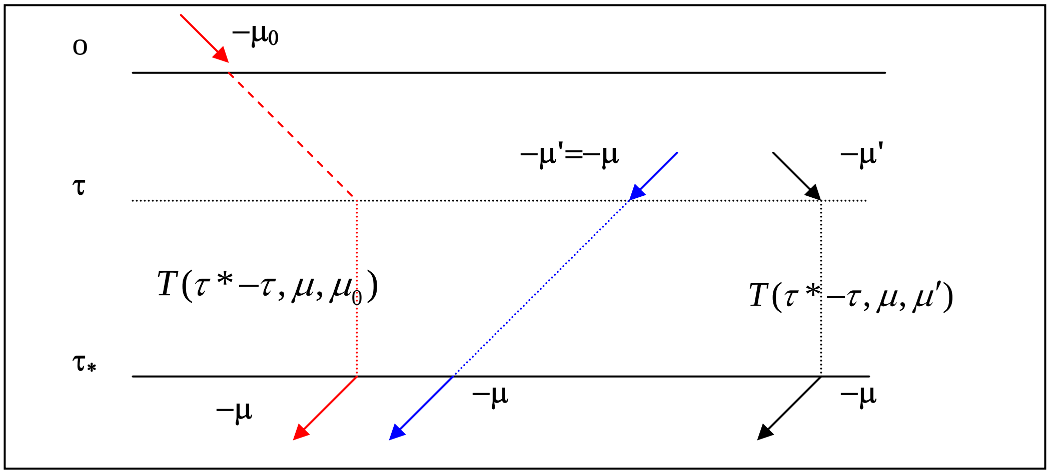
(3) The reflected (upward) intensity at the top of the finite atmosphere (τ = 0) is equivalent to (a) the reflection of solar flux plus (b) the direct and diffuse transmission of the upward diffuse intensity above the level τ:

$$I^{\uparrow}(0,\mu) = \frac{\mu_0 F_0}{\pi} R(\tau,\mu,\mu_0) + 2\int_0^1 T(\tau,\mu,\mu')I^{\uparrow}(\tau,\mu')\mu' d\mu' + I^{\uparrow}(\tau,\mu)\exp(-\tau/\mu) \qquad [21.21]$$



(4) The diffusely transmitted (downward) intensity at the bottom of the finite atmosphere (τ=τ*) is equivalent to (a) the transmission of the attenuated solar flux at the level τ plus (b) the direct and diffuse transmission of the downward diffuse intensity at the level τ from above:

$$I^{\downarrow}(\tau^*,-\mu) = \frac{\mu_0 F_0}{\pi}\exp(-\tau/\mu_0)T(\tau^*-\tau,\mu,\mu_0) + 2\int_0^1 T(\tau^*-\tau,\mu,\mu')I^{\downarrow}(\tau,-\mu')\mu' d\mu' + \qquad [21.22]$$
$$+ I^{\downarrow}(\tau,-\mu)\exp(-(\tau^*-\tau)/\mu)$$



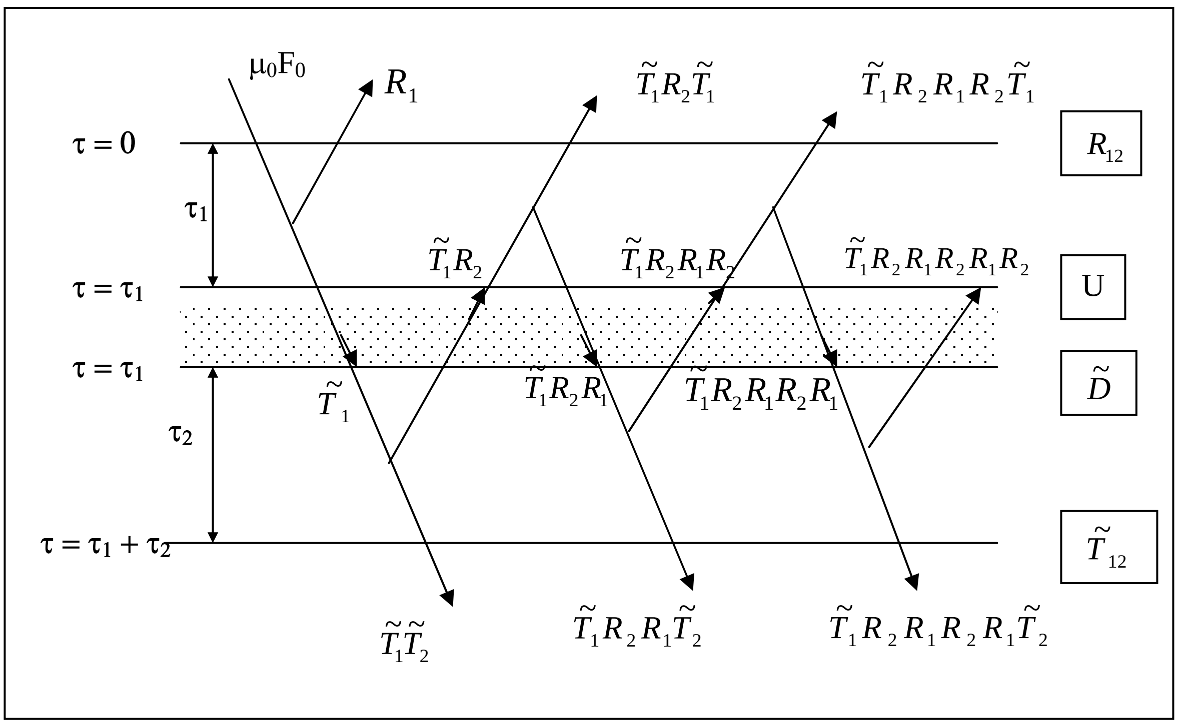
5. Adding method.

Adding method is an "exact" method for solving the radiative transfer equation with multiple scattering. It uses geometrical ray-tracing approach and the reflection and transmission of each individual atmospheric layer.

Strategy: knowing the reflection and transmission of two individual layers, the reflection and transmission of the combined layer may be obtained by calculating the successive reflections and transmissions between these two layers.

NOTE: If optical depths of these two layers are equaled, this method is referred to as the doubling-adding method.

Consider two layers with reflection R_1 and R_2 and total (direct plus diffuse) transmission $\tilde{T}_1$ and $\tilde{T}_2$ functions, respectively. Let's denote the combined reflection and total transmission functions by R_{12} and $\tilde{T}_{12}$, and combined reflection and total transmission functions between layers 1 and 2 by U and $\tilde{D}$, respectively.



The combined reflection function R_{12} is

$$R_{12} = R_1 + \tilde{T}_1 R_2 \tilde{T}_1 + \tilde{T}_1 R_2 R_1 R_2 \tilde{T}_1 + \tilde{T}_1 R_2 R_1 R_2 R_1 R_2 \tilde{T}_1 + \ldots =$$
$$= R_1 + \tilde{T}_1 R_2 \tilde{T}_1 [1 + R_1 R_2 + (R_1 R_2)^2 + \ldots] = \quad [21.23]$$
$$= R_1 + R_2 \tilde{T}_1^{\,2} (1 - R_1 R_2)^{-1}$$

NOTE: In Eq.[21.23] we use that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

The combined total transmission function $\tilde{T}_{12}$ is

$$\tilde{T}_{12} = \tilde{T}_1 + \tilde{T}_1 R_2 R_1 \tilde{T}_2 + \tilde{T}_1 R_2 R_1 R_2 R_1 \tilde{T}_2 + \ldots =$$
$$= \tilde{T}_1 \tilde{T}_2 [1 + R_1 R_2 + (R_1 R_2)^2 + \ldots] = \quad [21.24]$$
$$= \tilde{T}_1 \tilde{T}_2 (1 - R_1 R_2)^{-1}$$

The combined reflection function U between layers 1 and 2:

$$U = \tilde{T}_1 R_2 + \tilde{T}_1 R_2 R_1 R_2 + \tilde{T}_1 R_2 R_1 R_2 R_1 R_2 + \ldots =$$
$$= \tilde{T}_1 R_2 [1 + R_1 R_2 + (R_1 R_2)^2 + \ldots] = \quad [21.25]$$
$$= \tilde{T}_1 R_2 (1 - R_1 R_2)^{-1}$$

The combined total transmission function $\tilde{D}$ between layers 1 and 2:

$$\tilde{D} = \tilde{T}_1 + \tilde{T}_1 R_2 R_1 + \tilde{T}_1 R_2 R_1 R_2 R_1 + \ldots =$$
$$= \tilde{T}_1 [1 + R_1 R_2 + (R_1 R_2)^2 + \ldots] = \quad [21.26]$$
$$= \tilde{T}_1 (1 - R_1 R_2)^{-1}$$

From Eqs.[21.23]-[21.26], we find that

$$R_{12} = R_1 + \tilde{T}_1 U \; ; \quad \tilde{T}_{12} = \tilde{T}_2 \tilde{D} \; ; \quad U = R_2 \tilde{D} \quad [21.27]$$

Let's introduce $S = R_1 R_2 (1 - R_1 R_2)^{-1}$

Using that $\tilde{T} = T + \exp(-\tau / \mu')$, from Eqs.[21.26]-[21.27] we find

$$\tilde{D} = D + \exp(-\tau_1 / \mu_0) =$$
$$= (1+S)(T_1 + \exp(-\tau_1 / \mu_0)) = (1+S) T_1 + S \exp(-\tau_1 / \mu_0) + \exp(-\tau_1 / \mu_0) \quad [21.28]$$

$$\tilde{T}_{12} = (T_2 + \exp(-\tau_2/\mu_0))(D + \exp(-\tau_1/\mu_0))$$
$$= D\exp(-\tau_2/\mu_0) + T_2\exp(-\tau_1/\mu_0) + T_2 D + \exp\left(-\left[\frac{\tau_1}{\mu_0} + \frac{\tau_2}{\mu}\right]\right)\delta(\mu - \mu_0) \quad [21.29]$$

Thus, we can write a system of iterative equations for the computation of diffuse transmission and reflection for the two layers in the form:

$$\boxed{\begin{aligned} Q &= R_1 R_2 \\ S &= Q(1-Q)^{-1} \\ D &= T_1 + ST_1 + S\exp(-\tau_1/\mu_0) \\ U &= R_2 D + R_2\exp(-\tau_1/\mu_0) \\ R_{12} &= R_1 + \exp(-\tau_1/\mu)U + T_1 U \\ T_{12} &= \exp(-\tau_2/\mu)D + T_2\exp(-\tau_1/\mu_0) + T_2 D \end{aligned}} \quad [21.30]$$

NOTE: in Eq.[21.30], the product of two functions implies n integration over the appropriate angle so that all multiple-scattering contributions are included. For instance

$$R_1 R_2 = 2\int_0^1 R_1(\mu,\mu')R_2(\mu',\mu_0)\mu' d\mu'$$

Numerical procedure of the adding method:

1) As the starting point, one may calculate the reflection and transmission functions of an initial layer of very small optical depth (e.g., $\Delta\tau = 10^{-8}$) that the single scattering approximation is applicable.

2) Then, using Eq.[21.30], one computes the reflection and transmission functions of the layer of 2 $\Delta\tau$.

3) Using Eq.[21.30], one repeats the calculations adding the layers until a desirable optical depth is achieved.