Lecture 8.
Terrestrial infrared radiative processes. Part 2:

K-distribution approximation.

Objectives:
2. Correlated k-distribution approximation (CKD).

Required reading:
L02:4.3

Additional reading:

The KD method is developed to compute the spectral transmittance (hence the spectral intensity and spectral fluxes) based on grouping of gaseous absorption coefficients.

NOTE: The k-distribution approach was proposed by Ambartzumian in the 30th as an alternative to the computationally expensive line-by-line methods.

- The KD method benefits from the fact that the same value of $k_{a,v}$ is encountered many times over a given spectral interval => thus to eliminate the redundancy, one can group the values of $k_{a,v}$ and perform the transmittance calculation only once for a given value of $k_{a,v}$. 
**Strategy:**
Consider a **homogeneous** atmospheric layer. Spectral transmission is (see Lecture 7)

\[ T_\nu(u) = \frac{1}{\Delta \nu} \int \exp(-k_{a,\nu}u) d\nu \]

In the homogeneous atmospheric layer, spectral transmittance is independent of the ordering of \( k_{a,\nu} \) in a given spectral range, i.e., the order in which the wavenumbers are summed does not matter => so let’s sum them up from low to high \( k_{a,\nu} \).

Thus, we want to replace the integration over the wavenumber by integration over \( k \). It can be done by introducing a **normalized probability distribution function** for \( k_{a,\nu} \) (will be denoted by \( k \) below to simplified notation):

\[ T_\nu(u) = \frac{1}{\Delta \nu} \int \exp(-k,\nu) d\nu = \int \exp(-ku)f(k)dk \quad \text{[8.1]} \]

where we set \( \int_0^\infty f(k)dk = 1 \)

\( f(k) \) is the fraction of the spectral band with absorption coefficient between \( k \) and \( k+dk \).

\( f(k) \) is a smooth function.

---

**Figure 8.1 (a)** A schematic of absorption line spectra at two different pressure. **(b)** The two probability density function \( f(k) \) associated with (a). The shaded area shows the strongest absorption.
The **cumulative probability function** can be defined as

$$g(k) = \int_0^k f(k)\,dk \quad [8.2]$$

and $g(0)=0$; $g(\infty)=1$ and $dg(k)=f(k)\,dk$.

$g(k)$ is the fraction of the spectrum with absorption coefficient below $k$.

**NOTE:** By definition, $g(k)$ is a monotonically increasing and smooth function in $k$-space, Therefore, $k(g)$, as an inverse function of $g(k)$, is a smooth function in $g$-space.

Hence, the **spectral transmittance** can be written as

$$T_v(u) = \frac{1}{\Delta v} \int_{\Delta v} \exp(-k_{av}u)\,dv = \int_0^\infty \exp(-ku)f(k)\,dk = \int_0^1 \exp(-k(g)u)\,dg \quad [8.3]$$

**Figure 8.2 (a)** Absorption coefficient $k_{av}$ (in cm$^{-1}$ atm$^{-1}$) as a function of wavenumber in 9.6 µm ozone band (resolution of 0.05 cm$^{-1}$, $p=30$ mb, $T=200$K). **(b)** The probability density function $f(k)$ of the absorption coefficient. **(c)** The cumulative probability distribution function as a function of $k$. **(d)** Same as (c) but $k$ vs. $g$. 
Because both \( g(k) \) and \( k(g) \) are smooth functions, the above integral can be calculated by a finite sum as

\[
T_v(u) = \int_0^1 \exp(-k(g)u) dg \approx \sum_{i=1}^{N} \exp(-k(g_i)u) \Delta g_i =
\]

\[
= \Delta g_1 e^{-k_1 u} + \Delta g_2 e^{-k_2 u} + \ldots + \Delta g_N e^{-k_N u}
\]

where \( \Delta g_i \) is the quadrature weight.

Thus, the KD method allows calculating the spectral transmittance as a finite weighted sum of exponent in g-space, replacing the tedious wavenumber integration that is required in the line-by-line computations.

**Numerical realization of the KD method:**

(see illustration below)

Consider a spectral interval \( \Delta \nu \) that contains numerous absorption lines.

Let’s divide it into \( N \) intervals of \( \Delta \nu_j, j = 1,2,3\ldots N \)

The probability distribution function can be written as

\[
f(k) = \frac{1}{\Delta \nu} \frac{d \nu}{dk} = \frac{1}{\Delta \nu} \sum_j \frac{\Delta \nu_j}{\Delta k}
\]

where \( \Delta \nu_j \) is the subinterval of \( \Delta \nu \) where \( k \) is a monotonic function of \( \nu \).

Then the cumulative probability is

\[
g(k) = \frac{1}{\Delta \nu} \sum_j \int_0^k \frac{\Delta \nu_j}{\Delta k} \, dk' = \frac{1}{\Delta \nu} \sum_j \int_0^k \Delta \nu_j(k) = \frac{n(0,k)}{N}
\]

where \( n(0,k) \) is the number of computational points that contribute to \( k \) cumulatively.
Figure 8.3 How to calculate the absorption coefficient in g-space from the known absorption coefficient in the wavenumber domain. Solid line gives absorption coefficient as a function of \( \nu \). Numbers on the right side are the data points in each \( \Delta k \) interval (total number \( N=35 \)).

Thus by definition, \( g(j\Delta k)=n(0, j\Delta k)/N \)

CKD is the extension of KD for an inhomogeneous atmosphere.

Each pressure and temperature along the path has a unique $k_{a,v}$ spectrum.

The CKD method sorts each $k_{a,v}(p, T)$ spectrum independently to make k-distributions $k(g, p, T)$ for each $p$ and $T$.

NOTE: In practice, discrete k-distributions $k_j(p_l, T_m)$ are made for a set of pressures $p_l$ and temperatures $T_m$ and interpolated to any $p$ and $T$.

Overlap of gases in spectral band:

What do we do about multiple gases absorbing in one spectral band?

Overlap method #1: assumes that absorption spectra are independent

$$
\int T_v^{(1)} T_v^{(2)} d\nu = T_p^{(1)} T_p^{(2)} = \sum_{i=1}^{N} \exp(-k_i(g_i)u_i) \Delta g_{1,i} \sum_{j}^{M} \exp(-k_j(g_j)u_j) \Delta g_{2,j}
$$

$$
T_p(u_1, u_2) = \sum_{i=1}^{N} \Delta g_{1,i} \sum_{j}^{M} \Delta g_{2,j} \exp(-\tau_{mn})
$$

where $\tau_{mn} = k_{1m}u_1 + k_{2n}u_2$,

thus we have $M \times N$ terms

Overlap method #2: introduces the mixing ratio as an additional factor, so $k(g, p, T, q)$
**Examples of correlated k-distribution radiative transfer codes:**

**Fu&Liou code:**


Fu&Liou code is available on-line at [http://snowdog.larc.nasa.gov/cgi-bin/rose/flp200503/flp200503.cgi](http://snowdog.larc.nasa.gov/cgi-bin/rose/flp200503/flp200503.cgi)

You may get a free copy of the code or run it on-line by providing the input in the ther-user-friendly window.

Fu&Liou code divides shortwave into 6 bands with total 54 $k$’s and longwave into 12 bands with total of 121 $k$’s.

<table>
<thead>
<tr>
<th>Band</th>
<th>Region (cm$^{-1}$)</th>
<th>$N$ g’s</th>
<th>Gases</th>
<th>Solar Flux (W/m$^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>50000-14500</td>
<td>10</td>
<td>O3</td>
<td>619.62</td>
</tr>
<tr>
<td>2</td>
<td>14500-7700</td>
<td>8</td>
<td>H2O</td>
<td>484.30</td>
</tr>
<tr>
<td>3</td>
<td>7700-5250</td>
<td>12</td>
<td>H2O</td>
<td>149.85</td>
</tr>
<tr>
<td>4</td>
<td>5250-4000</td>
<td>7</td>
<td>H2O</td>
<td>48.73</td>
</tr>
<tr>
<td>5</td>
<td>4000-2850</td>
<td>12</td>
<td>H2O</td>
<td>31.66</td>
</tr>
<tr>
<td>6</td>
<td>2850-2500</td>
<td>5</td>
<td>H2O</td>
<td>5.80</td>
</tr>
<tr>
<td>7</td>
<td>2200-1900</td>
<td>2</td>
<td>H2O</td>
<td>Overlap method</td>
</tr>
<tr>
<td>8</td>
<td>1900-1700</td>
<td>3</td>
<td>H2O</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>1700-1400</td>
<td>4</td>
<td>H2O</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>1400-1250</td>
<td>4</td>
<td>H2O, CH4, N2O</td>
<td>1</td>
</tr>
<tr>
<td>11</td>
<td>1250-1100</td>
<td>3</td>
<td>H2O, CH4, N2O</td>
<td>1</td>
</tr>
<tr>
<td>12</td>
<td>1100-980</td>
<td>5</td>
<td>H2O, O3</td>
<td>1</td>
</tr>
<tr>
<td>13</td>
<td>980-800</td>
<td>2</td>
<td>H2O</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>800-670</td>
<td>10</td>
<td>H2O, CO2</td>
<td>2</td>
</tr>
<tr>
<td>15</td>
<td>670-540</td>
<td>12</td>
<td>H2O, CO2</td>
<td>2</td>
</tr>
<tr>
<td>16</td>
<td>540-400</td>
<td>7</td>
<td>H2O</td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>400-280</td>
<td>7</td>
<td>H2O</td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>280-0</td>
<td>8</td>
<td>H2O</td>
<td></td>
</tr>
</tbody>
</table>
**RRTM code:** [http://rtweb.aer.com/rrtm_frame.html](http://rtweb.aer.com/rrtm_frame.html)


**Table.** RRTM (version 3.1) Bands and Included Species

<table>
<thead>
<tr>
<th>Band</th>
<th>Wavenumber Range (cm⁻¹)</th>
<th>1050 - 96 mb</th>
<th>96 - 0.01 mb</th>
<th>Halocarbons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10 - 350</td>
<td>H₂O,N₂~</td>
<td>H₂O,N₂~</td>
<td>----</td>
</tr>
<tr>
<td>2</td>
<td>350 - 500</td>
<td>H₂O</td>
<td>H₂O</td>
<td>----</td>
</tr>
<tr>
<td>3</td>
<td>500 - 630</td>
<td>H₂O,CO₂,N₂O~</td>
<td>H₂O,CO₂,N₂O~</td>
<td>----</td>
</tr>
<tr>
<td>4</td>
<td>630 - 700</td>
<td>H₂O,CO₂</td>
<td>CO₂,O₃</td>
<td>----</td>
</tr>
<tr>
<td>5</td>
<td>700 - 820</td>
<td>H₂O,CO₂,O₃~</td>
<td>CO₂,O₃</td>
<td>CCL₄</td>
</tr>
<tr>
<td>6</td>
<td>820 - 980</td>
<td>H₂O,CO₂~</td>
<td>---</td>
<td>CFC11^,CFC12</td>
</tr>
<tr>
<td>7</td>
<td>980 - 1080</td>
<td>H₂O,O₃,CO₂~</td>
<td>O₃,CO₂</td>
<td>----</td>
</tr>
<tr>
<td>8</td>
<td>1080 - 1180</td>
<td>H₂O,CO₂~,O₃~,N₂O~</td>
<td>O₃,CO₂~,N₂O~</td>
<td>CFC12,CFC22^</td>
</tr>
<tr>
<td>9</td>
<td>1180 - 1390</td>
<td>H₂O,CH₄,N₂O~</td>
<td>CH₄,N₂O</td>
<td>----</td>
</tr>
<tr>
<td>10</td>
<td>1390 - 1480</td>
<td>H₂O</td>
<td>H₂O</td>
<td>----</td>
</tr>
<tr>
<td>11</td>
<td>1480 - 1800</td>
<td>H₂O,O₂~</td>
<td>H₂O,O₂~</td>
<td>----</td>
</tr>
<tr>
<td>12</td>
<td>1800 - 2080</td>
<td>H₂O,CO₂</td>
<td>---</td>
<td>----</td>
</tr>
<tr>
<td>13</td>
<td>2080 - 2250</td>
<td>H₂O,N₂O,CO₂~,CO~</td>
<td>O₃~</td>
<td>----</td>
</tr>
<tr>
<td>14</td>
<td>2250 - 2380</td>
<td>CO₂</td>
<td>CO₂</td>
<td>----</td>
</tr>
<tr>
<td>15</td>
<td>2380 – 2600</td>
<td>N₂O,CO₂,WV~,N₂~</td>
<td>---</td>
<td>----</td>
</tr>
<tr>
<td>16</td>
<td>2600 - 3250</td>
<td>H₂O,CH₄</td>
<td>CH₄</td>
<td>----</td>
</tr>
</tbody>
</table>

**Some features of RRTM LW:**

- k-distributions are obtained directly from a line-by-line radiative transfer code, LBLRTM.
- Fluxes and cooling rates can be calculated over sixteen bands in the longwave (10-3250 cm⁻¹).
- Modeled molecular absorbers are: water vapor, carbon dioxide, ozone, nitrous oxide, methane, oxygen, nitrogen and halocarbons.
- Fluxes calculated by RRTM agree with those computed by LBLRTM within 1.0 W/m² at all levels, and computed cooling rates agree to within 0.1 K/day in the troposphere and 0.3 K/day in the stratosphere.