Lecture 19.

1. Concepts of the direct and diffuse (scattered) solar radiation.
2. Source function and a radiative transfer equation for diffuse solar radiation.
4. Legendre polynomial expansion of the scattering phase function.

Required reading:
L02: 3.4, 6.1, Appendix E

1. Concepts of the direct and diffuse solar radiation.
   • The solar radiation field can be represented as a sum of two distinctly different components: direct and diffuse: \( I = I_{\text{dir}} + I_{\text{dif}} \)

Direct solar radiation is a component of solar radiation field that has survived the extinction passing a layer with optical depth \( \tau^* \) and it obeys the Beer-Bouguer-Lambert (extinction) law:

\[
I_{\text{dir}}^\perp = I_0 \exp(-\tau^*/\mu_0) \tag{19.1}
\]

where \( I_0 \) is the solar intensity at a given wavelength at the top of the atmosphere and \( \mu_0 \) is a cosine of the solar zenith angle \( \theta_0 \) (\( \mu_0 = \cos(\theta_0) \)).

The direct solar flux is

\[
F_{\text{dir}}^\perp = \mu_0 F_0 \exp(-\tau^*/\mu_0) \tag{19.2}
\]

2. Source function and a radiative transfer equation for the diffuse solar radiation.
**Diffuse radiation** arises from the light that undergoes one scattering event (**single scattering**) or many (**multiple scattering**).

The source function is defined as (see Lecture 3)

\[
J_\lambda = \left( j_{\lambda,\text{thermal}} + j_{\lambda,\text{scattering}} \right) / \beta_{e,\lambda}
\]

where \( j_{\lambda,\text{thermal}} \) is the **thermal emission** (\( j_{\lambda,\text{thermal}} = \beta_{a,\lambda} B_{\lambda}(T) \))

and \( j_{\lambda,\text{scattering}} \) is the radiation source from multiple scattering.

Using the volume scattering coefficient \( \beta_{s,\lambda} \) and the phase function \( P(\mu, \varphi, \mu',\varphi') \), we have

\[
j_{\lambda,\text{scattering}}(\tilde{\Omega}) = \frac{\beta_{s,\lambda}}{4\pi} \int_{\Omega'} I(\tilde{\Omega}') P(\tilde{\Omega}, \tilde{\Omega}') d\Omega'
\]  
[19.3]

**NOTE**: Recall the **scattering phase function** \( P(\mu, \varphi, \mu',\varphi') \) (i.e., the element of the scattering matrix \( P_{11} \)) represents the angular distribution of scattered energy as a function of direction. By the definition it is normalized as

\[
\frac{1}{4\pi} \int_{\Omega} P(\cos \Theta) d\Omega = 1
\]

where \( \Theta \) is the scattering angle

\[
cos(\Theta) = \cos(\Theta')\cos(\Theta) + \sin(\Theta')\sin(\Theta) \cos(\varphi' - \varphi) = \mu'\mu + (1 - \mu^2)^{1/2}(1 - \mu'^2)^{1/2} \cos(\varphi' - \varphi)
\]
Thus the source function for diffuse solar radiation may be written as two components

$$J(\tau, \mu, \varphi) = \frac{\alpha_0}{4\pi} \int \int I(\tau, \mu', \varphi') P(\mu, \varphi, \mu', \varphi') d\mu' d\varphi' + \frac{\alpha_0}{4\pi} F_0 P(\mu, \varphi, -\mu_0, \varphi_0) \exp(-\tau / \mu_0) \quad [19.4]$$

where the $\omega_0$ is the single scattering albedo and $P$ is the scattering phase function.

**NOTE:** In Eq.[19.4], the first term on the right-hand side shows that the phase function redirects the incoming intensity in the direction $(\mu', \varphi')$ to the direction $(\mu, \varphi)$, and the integrals account for all possible scattering events within the $4\pi$ solid angle.

- The source function for scattering Eq.[19.4] is more complicated than a thermal source function:
  (i) It involves conditions throughout the atmosphere, while the thermal source function depends on local conditions only;
  (ii) The phase function $P(\mu, \varphi, \mu', \varphi')$ may be a very complex function of the directions (and, in general, state of polarization).

Recall the radiative transfer equation for a plane-parallel atmosphere

$$\mu \frac{dI_\lambda(\tau; \mu; \varphi)}{d\tau} = I_\lambda(\tau; \mu; \varphi) - J_\lambda(\tau; \mu; \varphi)$$

Thus, using the source function for scattering, we can write the radiative transfer equation for the diffuse radiation as (omitting the subscript $\text{dif}$ in $I$)

$$\mu \frac{dI(\tau, \Omega)}{d\tau} = I(\tau, \Omega) - \frac{\alpha_0}{4\pi} \int I(\tau, \Omega') P(\Omega, \Omega') d\Omega' - \frac{\alpha_0}{4\pi} F_0 P(\Omega, -\Omega_0) \exp(-\tau / \mu_0) \quad [19.5]$$

**NOTE:** Eq.[19.5] is an integro-differential equation. To solve Eq.[19.5], one needs to know the scattering coefficient $\beta_{s,\lambda}$, absorption coefficient $\beta_{a,\lambda}$ and scattering phase function $P(\mu, \varphi, \mu', \varphi')$ as a function of wavelength in each atmospheric layer.
Eq.[19.5] can be simplified if there is no dependency on the azimuth angle. For azimuthally independent case, we may define the phase function as

\[
P(\mu, \mu') = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} P(\mu, \varphi, \mu', \varphi') d\varphi'
\]

[19.6]

Using Eq.[19.6], we can write the azimuthally independent radiative transfer equation for the diffuse radiation

\[
\frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - \frac{\omega_0}{2} \int_{\varphi=-1}^{1} I(\tau, \mu') P(\mu, \mu') d\mu' - \frac{\omega_0}{4\pi} F_0 P(\mu, -\mu_0) \exp(-\tau / \mu_0)
\]

[19.7]

➢ To find a solution of the radiative transfer equation for diffuse radiation (i.e., to solve Eq.[19.5] or [19.7] ), a number of approximate and “exact” techniques have been developed:

**Approximate methods (Lectures 19-20):**

i) Single scattering approximation (this lecture)

ii) Two-stream approximations

iii) Eddington and Delta-Eddington approximations

**“Exact” methods (Lecture 21):**

i) Discrete-ordinate technique

ii) Adding-doubling technique

iii) Monte-Carlo technique

If light has been scattered only once, the source function from Eq.[19.3] becomes

\[ J(\tau, \mu, \varphi) = \frac{\omega_0}{4\pi} F_0 P(\mu, \varphi, -\mu_0, \varphi_0) \exp(-\tau / \mu_0) \]  \[ 19.8 \]

and using the solution of the radiation transfer in a plane-parallel atmosphere bounded by two sides at \( \tau=0 \) and \( \tau=\tau^* \):

for upward intensity (called reflected intensity)

\[ I_\uparrow^\lambda(\tau; \mu, \varphi) = I_\uparrow^\lambda(\tau^*; \mu, \varphi) \exp\left(-\frac{\tau^* - \tau}{\mu}\right) \]

\[ + \frac{1}{\mu} \int_0^{\tau^*} \exp\left(-\frac{\tau' - \tau}{\mu}\right) J_\uparrow^\lambda(\tau'; \mu, \varphi) d\tau' \]

and downward intensity (called transmitted intensity)

\[ I_\downarrow^\lambda(\tau, -\mu, \varphi) = I_\downarrow^\lambda(0, -\mu, \varphi) \exp\left(-\frac{\tau}{\mu}\right) \]

\[ + \frac{1}{\mu} \int_0^{\tau} \exp\left(-\frac{\tau - \tau'}{\mu}\right) J_\downarrow^\lambda(\tau', -\mu, \varphi) d\tau' \]

we can write the solution for diffuse radiation in the case of single scattering as

\[ I_\uparrow^\lambda(\tau; \mu, \varphi) = I_\uparrow^\lambda(\tau^*; \mu, \varphi) \exp\left(-\frac{\tau^* - \tau}{\mu}\right) \]

\[ + \frac{1}{\mu} \omega_0 F_0 P(\mu, \varphi, -\mu_0, \varphi_0) \int_0^{\tau^*} \exp\left(-\frac{\tau' - \tau}{\mu} + \frac{\tau'}{\mu_0}\right) d\tau' \]  \[ 19.9a \]

\[ I_\downarrow^\lambda(\tau; -\mu, \varphi) = I_\downarrow^\lambda(0, -\mu, \varphi) \exp\left(-\frac{\tau}{\mu}\right) \]

\[ + \frac{1}{\mu} \omega_0 F_0 P(-\mu, \varphi, -\mu_0, \varphi_0) \int_0^{\tau} \exp\left(-\frac{\tau' - \tau}{\mu} + \frac{\tau'}{\mu_0}\right) d\tau' \]  \[ 19.9b \]

Assuming that there is no diffuse downward radiation at the top of the atmosphere

\[ I_\downarrow^\lambda(0, -\mu, \varphi) = 0 \]

and no upward diffuse radiation at the surface (i.e., no reflection from the surface)

\[ I_\uparrow^\lambda(\tau^*, \mu, \varphi) = 0 \]  \[ 19.10 \]
From Eq.[19.9a,b] for a finite atmosphere with optical depth $\tau^*$, we find the reflected and transmitted diffuse intensities

\[
I^\uparrow_\lambda (0, \mu, \varphi) = \frac{\omega_0 \mu_0 F_0}{4\pi (\mu + \mu_0)} P(\mu, \varphi, -\mu_0, \varphi_0) \left[ 1 - \exp\left(-\frac{\tau^*}{\mu} \left(\frac{1}{\mu} + \frac{1}{\mu_0}\right)\right) \right] \tag{19.11}
\]

and for $\mu$ is NOT equaled to $\mu_0$

\[
I^\downarrow_\lambda (\tau^*, -\mu, \varphi) = \frac{\omega_0 \mu_0 F_0}{4\pi (\mu - \mu_0)} P(-\mu, \varphi, -\mu_0, \varphi_0) \left[ \exp\left(-\frac{\tau^*}{\mu}\right) - \exp\left(-\frac{\tau^*}{\mu_0}\right) \right] \tag{19.12}
\]

and for $\mu=\mu_0$

\[
I^\downarrow_\lambda (\tau^*, -\mu_0, \varphi) = \frac{\omega_0 \tau^* F_0}{4\pi \mu_0} P(-\mu_0, \varphi_0, -\mu_0, \varphi_0) \left[ \exp\left(-\frac{\tau^*}{\mu_0}\right) \right] \tag{19.13}
\]

- In the case of single scattering, the diffuse intensities are directly proportional to the scattering phase function.

**NOTE:** If the atmosphere is optically thin (i.e., small optical depth $\tau^* < 1$), then

Eq.[19.11] simplifies to

\[
I^\uparrow (0, \mu, \varphi) = \frac{\omega_0}{4\pi} F_0 P(\Theta) \frac{\tau^*}{\mu} \tag{19.14}
\]

(called the single scattering approximation and often used in remote sensing).

### 4. Legendre polynomial expansion of the scattering phase function.

For many practical applications, the phase function must be numerically expanded in Legendre polynomials with a finite number of terms $N$ as

\[
P(\cos \Theta) = \sum_{l=0}^{N} \sigma^*_l P_l(\cos \Theta) \tag{19.14}
\]

where $\Theta$ is the scattering angle

\[
\cos(\Theta) = \cos(\Theta') \cos(\Theta) + \sin(\Theta') \sin(\Theta) \cos(\Phi' - \Phi) = \mu^* \mu + (1 - \mu^*)^{1/2} (1 - \mu^2)^{1/2} \cos(\Phi' - \Phi)
\]

and $\sigma^*_l$ is the expansion coefficients expressed as

\[
\sigma^*_l = \frac{2l + 1}{2} \int_{-1}^{1} P(\cos \Theta) P_l(\cos \Theta) d \cos(\Theta) , \quad l=0, 1, \ldots, N \tag{19.15}
\]
NOTE: Orthogonal properties of the Legendre polynomials:

\[ \int_{-1}^{1} P_k(\cos \Theta)P_l(\cos \Theta)d\cos(\Theta) = 0 \quad \text{for} \quad l \neq k \]

\[ \int_{-1}^{1} P_k(\cos \Theta)P_l(\cos \Theta)d\cos(\Theta) = \frac{2}{2l+1} \quad \text{for} \quad l = k \]

The first few Legendre polynomials are given by:

\[ P_0(x) = 1 \]
\[ P_1(x) = x \]
\[ P_2(x) = \frac{1}{2}(3x^2 - 1) \]
\[ P_3(x) = \frac{1}{2}(5x^3 - 3x) \]
\[ P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \]
\[ P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x) \]
Rewriting the radiative transfer equation in terms of associated Legendre polynomials:

Eq.[19.14] can be expressed in the terms of associated Legendre polynomials

\[ P(\mu, \varphi, \mu', \varphi') = \sum_{m=0}^{N} \sum_{l=m}^{N} \sigma_i^m P_i^m(\mu)P_i^m(\mu') \cos(m(\varphi' - \varphi)) \]  \[ 19.16 \]

where

\[ \sigma_i^m = (2 - \delta_{0,m})\sigma_i^*(l-m)! \quad (l=m, \ldots, N; \quad 0 \leq m \leq N) \]

and \( \delta_{0,m} \) is the Kronecker delta: \( \delta_{0,m} = 1 \) for \( m=0 \) and otherwise \( \delta_{0,m} = 0 \).

In similar manner, we may expand the diffuse intensity in the cosine series

\[ I(\tau, \mu, \varphi) = \sum_{m=0}^{N} I_i^m(\tau, \mu) \cos(m(\varphi_0 - \varphi)) \]  \[ 19.17 \]

Using Eqs.[19.16] and [19.17] and the orthogonality of the associated Legendre polynomials, the equation of the radiative transfer for the diffuse intensity (Eq.[19.7]) splits into \((N+1)\) independent equations in the form

\[ \mu \frac{dI_i^m(\tau, \mu)}{d\tau} = I_i^m(\tau, \mu) - (1 + \delta_{0,m}) \frac{\omega_0}{4} \sum_{l=m}^{N} \sigma_i^m P_i^m(\mu) \int_{-1}^{1} P_i^m(\mu')I_i^m(\tau, \mu')d\mu' - \frac{\omega_0}{4\pi} \sum_{l=m}^{N} \sigma_i^m P_i^m(\mu)P_i^m(-\mu_0)F_0 \exp(-\tau / \mu_0) \] \[ 19.18 \]

**m=0 => azimuthal independent case:**

From Eq.[18.16], the azimuth-independent phase function can be expressed as

\[ P(\mu, \mu') = \sum_{l=0}^{N} \sigma_i P_i(\mu)P_i(\mu') \] \[ 19.19 \]

For this case Eq.[19.18] simplifies to (omitting the superscript 0 for \( m=0 \))

\[ \mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - \frac{\omega_0}{2} \sum_{l=0}^{N} \sigma_i P_i(\mu) \int_{-1}^{1} P_i(\mu')I(\tau, \mu')d\mu' - \frac{\omega_0}{4\pi} \sum_{l=0}^{N} \sigma_i P_i(\mu)P_i(-\mu_0)F_0 \exp(-\tau / \mu_0) \] \[ 19.20 \]
EXAMPLE: Expansion of the Henyey-Greenstein scattering phase function in the Legendre polynomials

The Henyey-Greenstein scattering phase function is a model phase function, which is often used in radiative transfer calculations:

$$P_{HG} (\cos \Theta) = \frac{1 - g^2}{(1 + g^2 - 2g \cos \Theta)^{3/2}}$$

g is the asymmetry parameter.
Let’s take $g=0.5$ (representative of aerosols)

Legendre expansion:

$$P_{HG} (\cos \Theta) = \sum_{l=0}^{N} \omega^*_l P_l (\cos \Theta) \quad \text{and}$$

$$\omega^*_l = \frac{2l + 1}{2} \int_{-1}^{1} P_{HG} (\cos \Theta) P_l (\cos \Theta) d \cos (\Theta)$$

If $N=0$ =>

$$\omega^*_0 = \frac{1}{2} \int_{-1}^{1} P_{HG} (\cos \Theta) P_0 (\cos \Theta) d \cos (\Theta) = \frac{1}{2} \int_{-1}^{1} P_{HG} (\cos \Theta) \ 1 \ d \cos (\Theta) = \frac{1}{2} \ 2 = 1$$

NOTE: $\omega^*_0$ is always 1 because of normalization of the phase function

$$\frac{1}{2} \int_{-1}^{1} P(\cos \Theta) d \cos (\Theta) = 1$$

Thus, in the case of one term in the expansion, we have

$$P_{HG} (\cos \Theta) \approx \sum_{l=0}^{0} \omega^*_l P_l (\cos \Theta) = 1 * 1 = 1$$

Plot (below) shows that using only one term gives poor approximation

NOTE: In all plots below, the black curve shows the Henyey-Greenstein phase function, and the red curve denotes the Legendre expansion, all as a function of $\cos (\Theta)$. 
If N=1 =>
\[ \sigma_o^* = \frac{1}{2} \int_{-1}^{1} P_{HG}(\cos \Theta) P_0(\cos \Theta) d \cos(\Theta) = \frac{1}{2} \int_{-1}^{1} P_{HG}(\cos \Theta) \ 1 \ d \cos(\Theta) = \frac{1}{2} \ 2 = 1 \]
\[ \sigma_1^* = \frac{1}{2} \int_{-1}^{1} P_{HG}(\cos \Theta) P_1(\cos \Theta) d \cos(\Theta) = \frac{1}{2} \int_{-1}^{1} P_{HG}(\cos \Theta) \ (\cos \Theta) \ d \cos(\Theta) = 1.5 \]
Thus, in the case of two terms in the expansion, we have
\[ P_{HG}(\cos \Theta) \approx \sum_{l=0}^{1} \sigma_l^* P_l(\cos \Theta) = 1 + 1.5(\cos \Theta) \quad \text{still accuracy is not so good!} \]
We can continue by including more terms to get desirable accuracy.
N=7-10 gives good approximation to the Henyey-Greenstein scattering phase function with g=0.5.

**NOTE:** The larger the asymmetry parameter g the larger number of terms will be required to achieve acceptable accuracy.