Lecture 19.

Principles of multiple scattering in the atmosphere. Radiative transfer equation for diffuse solar radiation. Single scattering approximation.

1. Concepts of the direct and diffuse (scattered) solar radiation.
2. Source function and a radiative transfer equation for diffuse solar radiation.
3. Single scattering approximation.
4. Legendre polynomial expansion of the scattering phase function.

Required reading:

L02: 3.4, 6.1, Appendix E

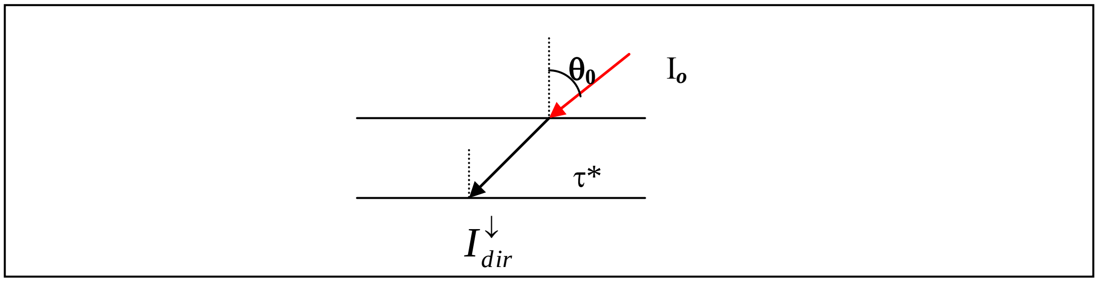
1. Concepts of the direct and diffuse solar radiation.

- The solar radiation field can be represented as a sum of two distinctly different components: **direct** and **diffuse:** $I = I_{dir} + I_{dif}$

Direct solar radiation is a component of solar radiation field that has survived the extinction passing a layer with optical depth **τ*** and it obeys the Beer-Bouguer-Lambert (extinction) law:

$$\boxed{I_{dir}^{\downarrow} = I_0 \exp(-\tau^* / \mu_0)} \quad [19.1]$$

where I_0 is the solar intensity at a given wavelength at the top of the atmosphere and μ_0 is a cosine of the solar zenith angle θ_0 ($\mu_0 = \cos(\theta_0)$).

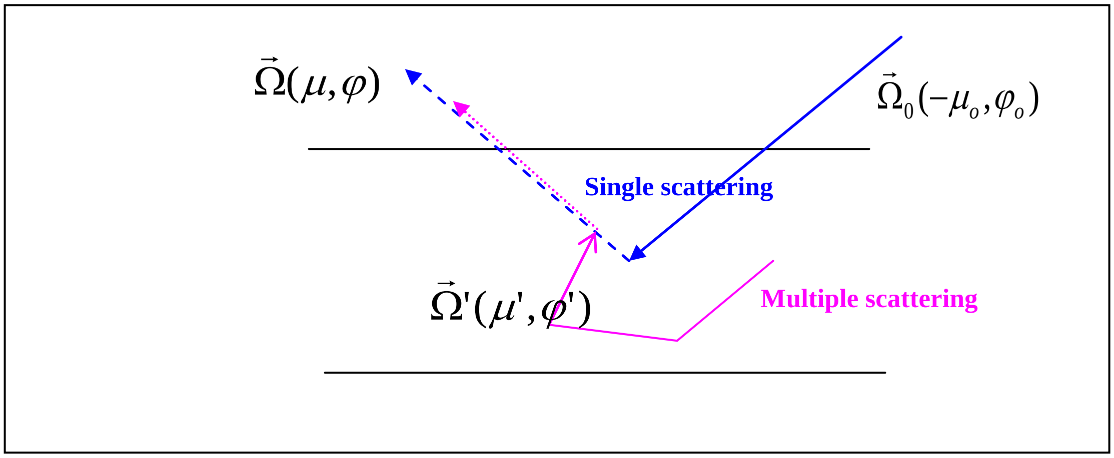


The **direct solar flux** is

$$\boxed{F_{dir}^{\downarrow} = \mu_0 F_0 \exp(-\tau^* / \mu_0)} \quad [19.2]$$

2. Source function and a radiative transfer equation for the diffuse solar radiation.

Diffuse radiation arises from the light that undergoes one scattering event (**single scattering**) or many (**multiple scattering**).



The source function is defined as (see Lecture 3)

$$J_\lambda = (\mathbf{j}_{\lambda,\ \mathrm{thermal}} + \mathbf{j}_{\lambda,\ \mathrm{scattering}}) / \beta_{e,\lambda}$$

where $\mathbf{j}_{\lambda,\ \mathrm{thermal}}$ is the **thermal emission (** $j_{\lambda,thermal} = \beta_{a,\lambda} B_\lambda(T)$)

and $\mathbf{j}_{\lambda,\ \mathrm{scattering}}$ is the radiation source from multiple scattering.

Using the volume scattering coefficient $\beta_{s,\lambda}$ and the phase function ***P*(μ, φ, μ',φ')**, we have

$$\boxed{j_{\lambda,scattering}(\vec{\Omega}) = \frac{\beta_{s,\lambda}}{4\pi} \int_{\vec{\Omega}'} I(\vec{\Omega}')P(\vec{\Omega},\vec{\Omega}')d\Omega'} \qquad [19.3]$$

NOTE: Recall the **scattering phase function *P*(μ, φ, μ',φ')** (i.e., the element of the scattering matrix P_{11}) represents the angular distribution of scattered energy as a function of direction. By the definition it is normalized as

$$\frac{1}{4\pi}\int_{\Omega} P(\cos\Theta)d\Omega = 1$$

where **Θ** is the scattering angle

$$\boldsymbol{cos(\Theta) = cos(\theta')cos(\theta) + sin(\theta')sin(\theta)\ cos(\varphi'-\varphi) = \mu'\mu + (1-\mu'^2)^{1/2}(1-\mu^2)^{1/2}\ cos(\varphi'-\varphi)}$$

Thus the **source function for diffuse solar radiation** may be written as two components

$$\boxed{\begin{aligned} J(\tau,\mu,\varphi) &= \frac{\omega_0}{4\pi}\int_0^{2\pi}\int_{-1}^{1} I(\tau,\mu',\varphi')P(\mu,\varphi,\mu',\varphi')d\mu' d\varphi' + \\ &+ \frac{\omega_0}{4\pi}F_0 P(\mu,\varphi,-\mu_0,\varphi_0)\exp(-\tau/\mu_0) \end{aligned}} \quad [19.4]$$

where the ω_0 is the single scattering albedo and ***P*** is the scattering phase function.

NOTE: In Eq.[19.4], the first term on the right-hand side shows that the phase function redirects the incoming intensity in the direction (**μ',φ'**) to the direction (**μ, φ**), and the integrals account for all possible scattering events within the 4π solid angle.

- The **source function for scattering** Eq.[19.4] is more complicated than a thermal source function:
  (i) It involves conditions throughout the atmosphere, while the thermal source function depends on local conditions only;
  (ii) The phase function ***P*(μ, φ, μ',φ')** may be a very complex function of the directions (and, in general, state of polarization).

Recall the radiative transfer equation for a plane-parallel atmosphere

$$\mu\frac{dI_\lambda(\tau;\mu;\varphi)}{d\tau} = I_\lambda(\tau;\mu;\varphi) - J_\lambda(\tau;\mu;\varphi)$$

Thus, using the source function for scattering, we can write the **radiative transfer equation for the <u>diffuse</u> radiation** as (omitting the subscript *dif* in ***I***)

$$\boxed{\mu\frac{dI(\tau,\vec{\Omega})}{d\tau} = I(\tau,\vec{\Omega}) - \frac{\omega_0}{4\pi}\int_{4\pi} I(\tau,\vec{\Omega}')P(\vec{\Omega},\vec{\Omega}')d\Omega' - \frac{\omega_0}{4\pi}F_0 P(\vec{\Omega},-\vec{\Omega}_0)\exp(-\tau/\mu_0)} \quad [19.5]$$

NOTE: Eq.[19.5] is an integro-differential equation. To solve Eq.[19.5], one needs to know the scattering coefficient $\beta_{s,\lambda}$, absorption coefficient $\beta_{a,\lambda}$ and scattering phase function ***P*(μ, φ, μ',φ')** as a function of wavelength in each atmospheric layer.

Eq.[19.5] can be simplified if there is no dependency on the azimuth angle.

For azimuthally independent case, we may define the phase function as

$$P(\mu,\mu') = \frac{1}{2\pi}\int_0^{2\pi} P(\mu,\varphi,\mu',\varphi')d\varphi' \qquad [19.6]$$

Using Eq.[19.6], we can write **the azimuthally independent radiative transfer equation for the diffuse radiation**

$$\boxed{\mu\frac{dI(\tau,\mu)}{d\tau} = I(\tau,\mu) - \frac{\omega_0}{2}\int_{-1}^{1} I(\tau,\mu')P(\mu,\mu')d\mu' - \frac{\omega_0}{4\pi}F_0 P(\mu,-\mu_0)\exp(-\tau/\mu_0)} \qquad [19.7]$$

- **To find a solution of the radiative transfer equation for diffuse radiation** (i.e., to solve Eq.[19.5] or [19.7]) , a number of approximate and "exact" techniques have been developed:

  Approximate methods (Lectures 19-20):

  i) Single scattering approximation (this lecture)

  ii) Two-stream approximations

  iii) Eddington and Delta-Eddington approximations

  "Exact" methods (Lecture 21):

  i) Discrete-ordinate technique

  ii) Adding-doubling technique

  iii) Monte-Carlo technique

3. Single scattering and the single scattering approximation.

If light has been scattered only once, the source function from Eq.[19.3] becomes

$$J(\tau,\mu,\varphi) = \frac{\omega_0}{4\pi} F_0 P(\mu,\varphi,-\mu_0,\varphi_0)\exp(-\tau/\mu_0) \qquad [19.8]$$

and using the solution of the radiation transfer in a plane-parallel atmosphere bounded by two sides at τ=0 and τ=τ*:

for upward intensity (called reflected intensity)

$$I_\lambda^{\uparrow}(\tau;\mu;\varphi) = I_\lambda^{\uparrow}(\tau^*;\mu;\varphi)\exp\left(-\frac{\tau^*-\tau}{\mu}\right) + \frac{1}{\mu}\int_{\tau}^{\tau^*} \exp\left(-\frac{\tau'-\tau}{\mu}\right) J_\lambda^{\uparrow}(\tau';\mu;\varphi)d\tau'$$

and downward intensity (called transmitted intensity)

$$I_\lambda^{\downarrow}(\tau,-\mu,\varphi) = I_\lambda^{\downarrow}(0,-\mu,\varphi)\exp\left(-\frac{\tau}{\mu}\right) + \frac{1}{\mu}\int_{0}^{\tau} \exp\left(-\frac{\tau-\tau'}{\mu}\right) J_\lambda^{\downarrow}(\tau',-\mu,\varphi)d\tau'$$

we can write **the solution for diffuse radiation in the case of single scattering** as

$$I_\lambda^{\uparrow}(\tau;\mu;\varphi) = I_\lambda^{\uparrow}(\tau^*,\mu,\varphi)\exp\left(-\frac{\tau^*-\tau}{\mu}\right) + \frac{1}{\mu}\frac{\omega_0}{4\pi}F_0 P(\mu,\varphi,-\mu_0,\varphi_0)\int_{\tau}^{\tau^*}\exp\left(-\left[\frac{\tau'-\tau}{\mu}+\frac{\tau'}{\mu_0}\right]\right)d\tau' \qquad [19.9a]$$

$$I_\lambda^{\downarrow}(\tau;-\mu;\varphi) = I_\lambda^{\downarrow}(0,-\mu,\varphi)\exp\left(-\frac{\tau}{\mu}\right) + \frac{1}{\mu}\frac{\omega_0}{4\pi}F_0 P(-\mu,\varphi,-\mu_0,\varphi_0)\int_{0}^{\tau}\exp\left(-\left[\frac{\tau'-\tau}{\mu}+\frac{\tau'}{\mu_0}\right]\right)d\tau' \qquad [19.9b]$$

Assuming that there is no diffuse downward radiation at the top of the atmosphere

$$I^{\downarrow}(0,-\mu,\varphi) = 0$$

and no upward diffuse radiation at the surface (i.e., no reflection from the surface)

$$I^{\uparrow}(\tau^*,\mu,\varphi) = 0 \qquad [19.10]$$

From Eq.[19.9a,b] for a finite atmosphere with optical depth τ*, we find the **reflected and transmitted diffuse intensities**

$$I_\lambda^\uparrow(0,\mu,\varphi) = \frac{\omega_0 \mu_0 F_0}{4\pi(\mu+\mu_0)} P(\mu,\varphi,-\mu_0,\varphi_0)\left[1-\exp\left(-\tau^*(\frac{1}{\mu}+\frac{1}{\mu_0})\right)\right] \quad [19.11]$$

and for **μ** is NOT equaled to $\boldsymbol{\mu_0}$

$$I_\lambda^\downarrow(\tau^*,-\mu,\varphi) = \frac{\omega_0 \mu_0 F_0}{4\pi(\mu-\mu_0)} P(-\mu,\varphi,-\mu_0,\varphi_0)\left[\exp\left(-\frac{\tau^*}{\mu}\right)-\exp\left(-\frac{\tau^*}{\mu_0}\right)\right] \quad [19.12]$$

and for $\boldsymbol{\mu=\mu_0}$

$$\boldsymbol{I_\lambda^\downarrow(\tau^*,-\mu_0,\varphi) = \frac{\omega_0 \tau^* F_0}{4\pi\mu_0} P(-\mu_0,\varphi_0,-\mu_0,\varphi_0)\left[\exp\left(-\frac{\tau^*}{\mu_0}\right)\right]} \quad [19.13]$$

- In the case of single scattering, the diffuse intensities are directly proportional to the scattering phase function.

NOTE: If the atmosphere is optically thin (i.e., small optical depth τ* <1), then

Eq.[19.11] simplifies to
$$\boxed{I^\uparrow(0,\mu,\varphi) = \frac{\omega_0}{4\pi} F_0 P(\Theta)\frac{\tau^*}{\mu}}$$

(called the single scattering approximation and often used in remote sensing).

<u>4. Legendre polynomial expansion of the scattering phase function.</u>

For many practical applications, the phase function must be numerically expanded in Legendre polynomials with a finite number of terms N as

$$P(\cos\Theta) = \sum_{l=0}^{N} \varpi_l^* P_l(\cos\Theta) \quad [19.14]$$

where **Θ** is the scattering angle

***cos*(Θ) = *cos*(θ')*cos*(θ) + *sin*(θ')*sin*(θ) *cos*(φ'-φ) = μ'μ + (1–μ'²)^1/2(1–μ²)^1/2 *cos*(φ'-φ)**

and ϖ_l^* is the expansion coefficients expressed as

$$\varpi_l^* = \frac{2l+1}{2}\int_{-1}^{1} P(\cos\Theta)P_l(\cos\Theta)d\cos(\Theta)\,,\ l=0,1,\ldots,N \quad [19.15]$$

NOTE: Orthogonal properties of the Legendre polynomials:

$$\int_{-1}^{1} P_k(\cos\Theta)P_l(\cos\Theta)d\cos(\Theta) = 0 \; for \;\; l \neq k$$

$$\int_{-1}^{1} P_k(\cos\Theta)P_l(\cos\Theta)d\cos(\Theta) = \frac{2}{2l+1} \; for \;\; l = k$$

The first few Legendre polynomials are given by:

$$P_0(x) = 1$$

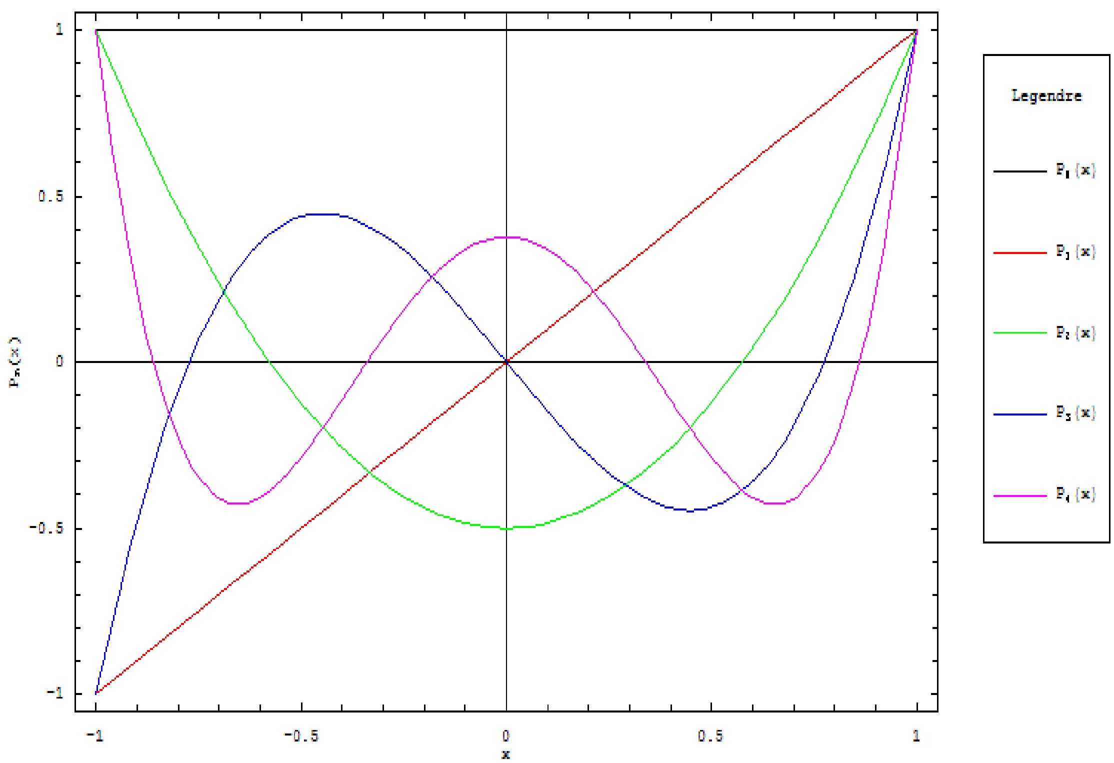
$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$



Rewriting the radiative transfer equation in terms of associated Legendre polynomials:

Eq.[19.14] can be expressed in the terms of associated Legendre polynomials

$$P(\mu,\varphi,\mu',\varphi') = \sum_{m=0}^{N} \sum_{l=m}^{N} \varpi_l^m P_l^m(\mu) P_l^m(\mu') \cos(m(\varphi'-\varphi)) \qquad [19.16]$$

where

$$\varpi_l^m = (2-\delta_{0,m})\varpi_l^* \frac{(l-m)!}{(l+m)!} \qquad \boldsymbol{l = m,\ldots,N;} \qquad 0 \le m \le N$$

and $\delta_{0,m}$ is the Kronecker delta: $\delta_{0,m} = 1$ for m=0 and otherwise $\delta_{0,m} = 0$.

In similar manner, we may expand the diffuse intensity in the cosine series

$$I(\tau,\mu,\varphi) = \sum_{m=0}^{N} I^m(\tau,\mu)\cos(m(\varphi_0-\varphi)) \qquad [19.17]$$

Using Eqs.[19.16] and [19.17] and the orthogonality of the associated Legendre polynomials, the equation of the radiative transfer for the diffuse intensity (Eq.[19.7]) splits into (N+1) independent equations in the form

$$\mu\frac{dI^m(\tau,\mu)}{d\tau} = I^m(\tau,\mu) - (1+\delta_{0,m})\frac{\omega_0}{4}\sum_{l=m}^{N} \varpi_l^m P_l^m(\mu)\int_{-1}^{1} P_l^m(\mu')I^m(\tau,\mu')d\mu' - $$
$$-\frac{\omega_0}{4\pi}\sum_{l=m}^{N}\varpi_l^m P_l^m(\mu)P_l^m(-\mu_o)F_0\exp(-\tau/\mu_0) \qquad [19.18]$$

m=0 <u>=> azimuthal independent case:</u>

From Eq.[18.16], the azimuth-independent phase function can be expressed as

$$P(\mu,\mu') = \sum_{l=0}^{N}\varpi_l P_l(\mu)P_l(\mu') \qquad [19.19]$$

For this case Eq.[19.18] simplifies to (omitting the superscript 0 for m=0)

$$\boxed{\begin{aligned}&\mu\frac{dI(\tau,\mu)}{d\tau} = I(\tau,\mu) - \frac{\omega_0}{2}\sum_{l=0}^{N} \varpi_l^* P_l(\mu)\int_{-1}^{1} P_l(\mu')I(\tau,\mu')d\mu' - \\ &-\frac{\omega_0}{4\pi}\sum_{l=0}^{N}\varpi_l^* P_l(\mu)P_l(-\mu_o)F_0\exp(-\tau/\mu_0)\end{aligned}} \qquad [19.20]$$

EXAMPLE: ***Expansion of the Henyey-Greenstein scattering phase function in the Legendre polynomials***

The Henyey-Greenstein scattering phase function is a model phase function, which is often used in radiative transfer calculations:

$$P_{HG}(\cos\Theta) = \frac{1-g^2}{(1+g^2-2g\cos\Theta)^{3/2}}$$

g is the asymmetry parameter.
Let's take g=0.5   (representative of aerosols)

Legendre expansion:

$$P_{HG}(\cos\Theta) = \sum_{l=0}^{N} \varpi_l^* P_l(\cos\Theta) \qquad \text{and}$$

$$\varpi_l^* = \frac{2l+1}{2}\int_{-1}^{1} P_{HG}(\cos\Theta)P_l(\cos\Theta)d\cos(\Theta)$$

If N=0 =>

$$\varpi_o^* = \frac{1}{2}\int_{-1}^{1} P_{HG}(\cos\Theta)P_0(\cos\Theta)d\cos(\Theta) = \frac{1}{2}\int_{-1}^{1} P_{HG}(\cos\Theta)\ 1\ d\cos(\Theta) = \frac{1}{2}\ 2 = 1$$

NOTE: ϖ_o^* is always 1 because of normalization of the phase function

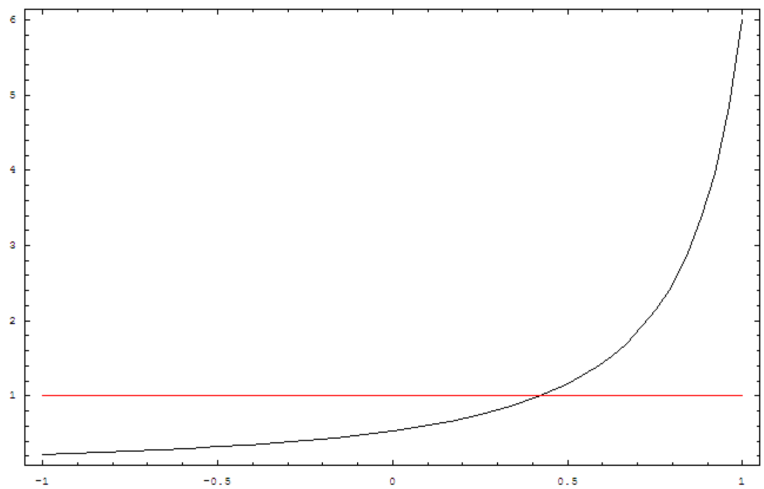
$$\frac{1}{2}\int_{-1}^{1} P(\cos\Theta)d\cos(\Theta) = 1$$

Thus, in the case of one term in the expansion, we have

$$P_{HG}(\cos\Theta) \approx \sum_{l=0}^{0} \varpi_l^* P_l(\cos\Theta) = 1*1 = 1$$

Plot (below) shows that using only one term gives poor approximation

NOTE: In all plots below, the black curve shows the Henyey-Greenstein phase function, and the red curve denotes the Legendre expansion, all as a function of cos(Θ).



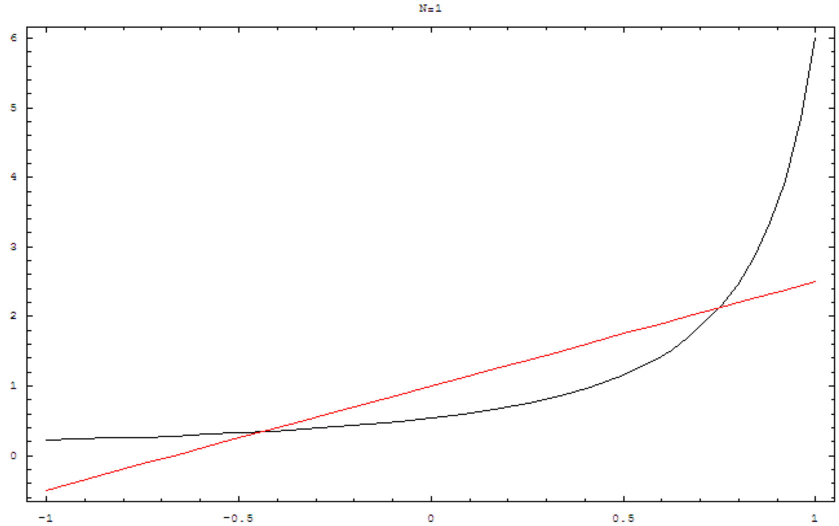
<u>If N=1</u> =>

$$\varpi_o^* = \frac{1}{2}\int_{-1}^{1} P_{HG}(\cos\Theta)P_0(\cos\Theta)d\cos(\Theta) = \frac{1}{2}\int_{-1}^{1} P_{HG}(\cos\Theta)\ 1\ d\cos(\Theta) = \frac{1}{2}\ 2 = 1$$

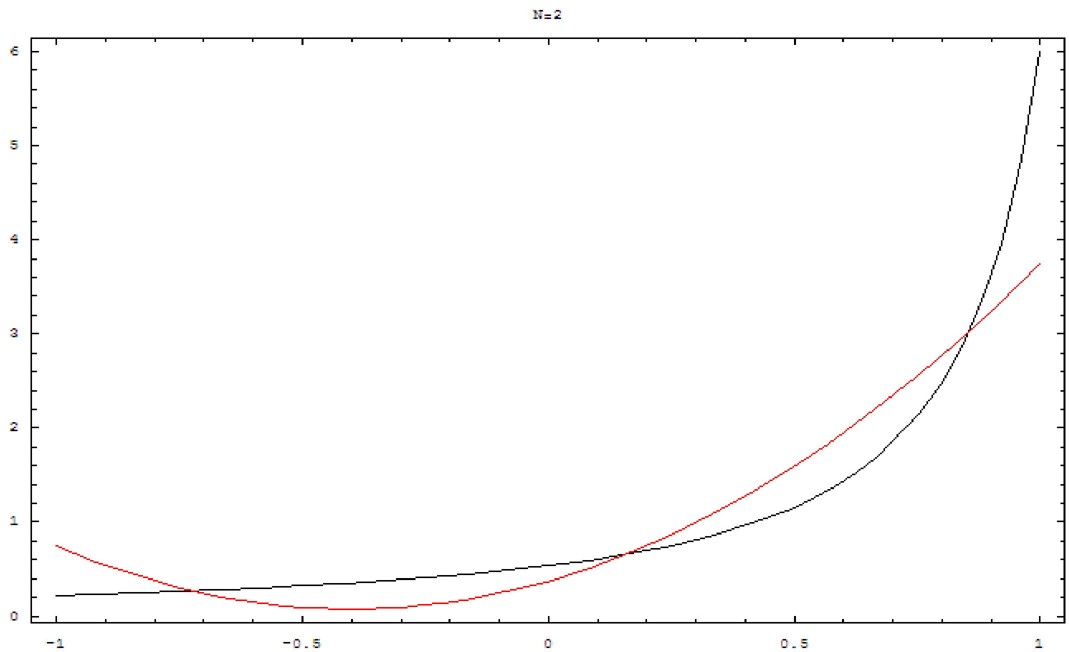
$$\varpi_1^* = \frac{1}{2}\int_{-1}^{1} P_{HG}(\cos\Theta)P_1(\cos\Theta)d\cos(\Theta) = \frac{1}{2}\int_{-1}^{1} P_{HG}(\cos\Theta)\ (\cos\Theta)\ d\cos(\Theta) = 1.5$$

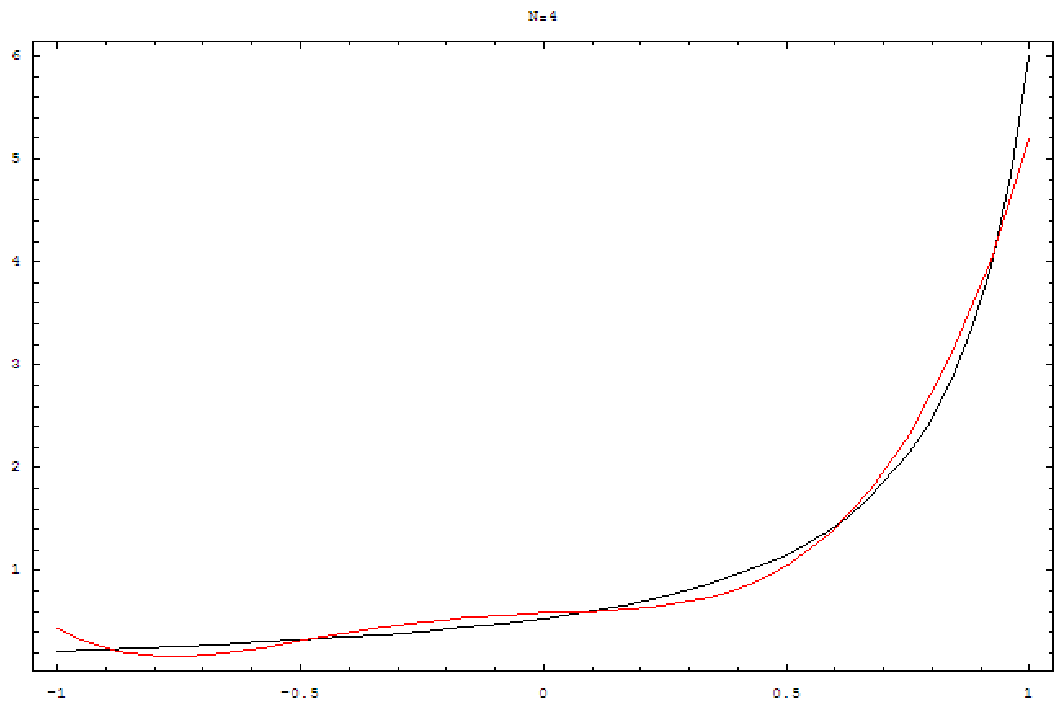
Thus, in the case of two terms in the expansion, we have

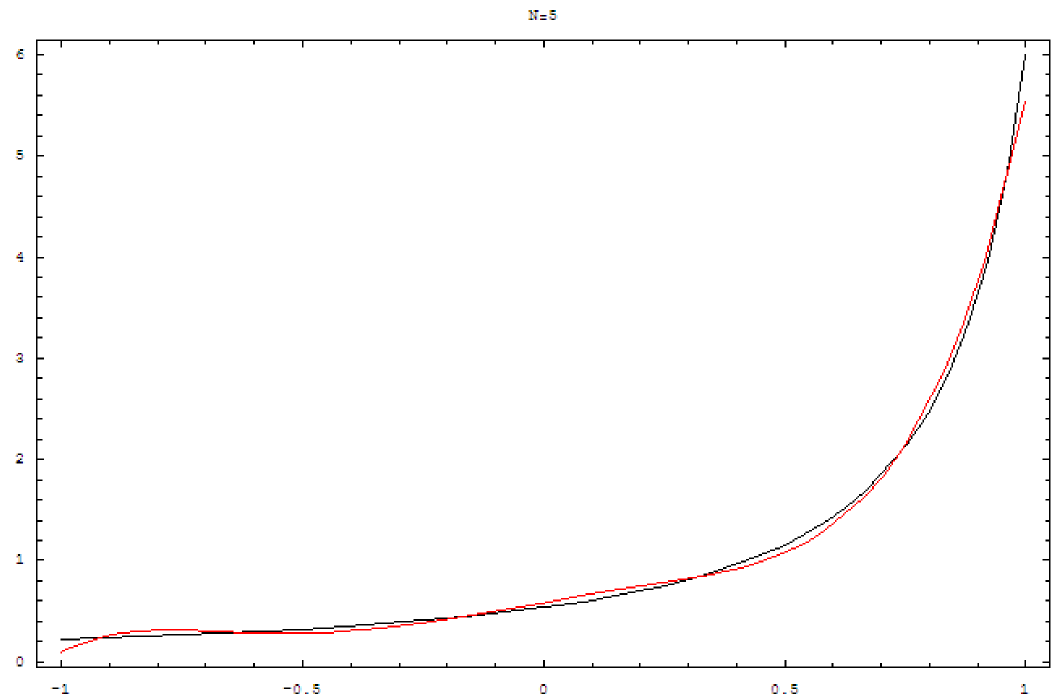
$$P_{HG}(\cos\Theta) \approx \sum_{l=0}^{1} \varpi_l^* P_l(\cos\Theta) = 1 + 1.5(\cos\Theta)$$ - still accuracy is not so good!

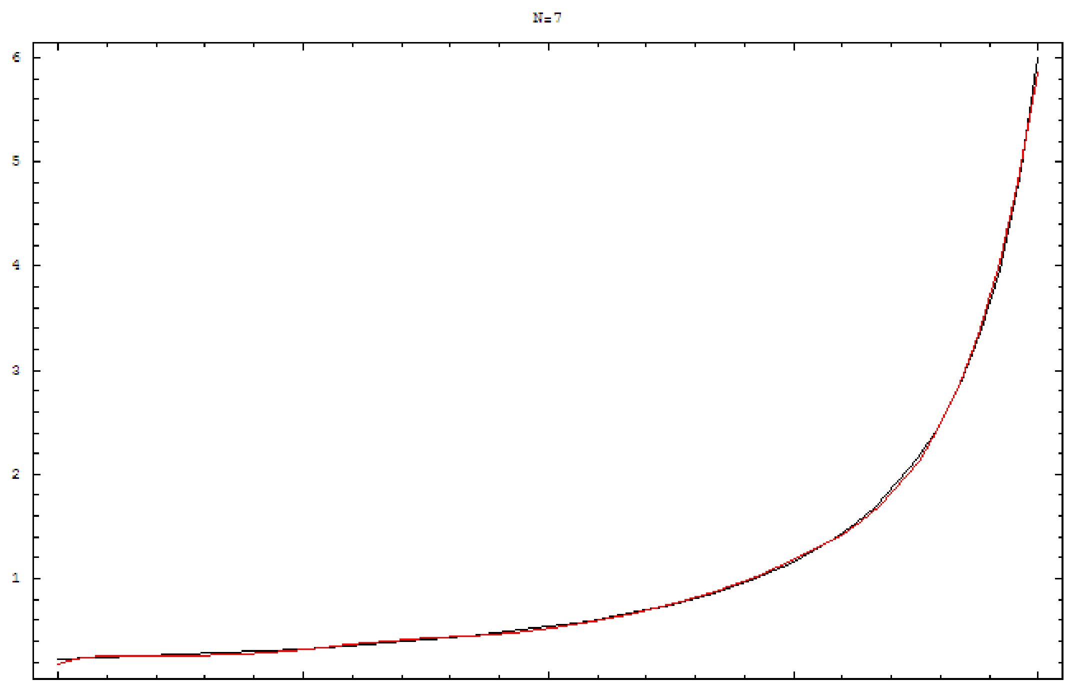


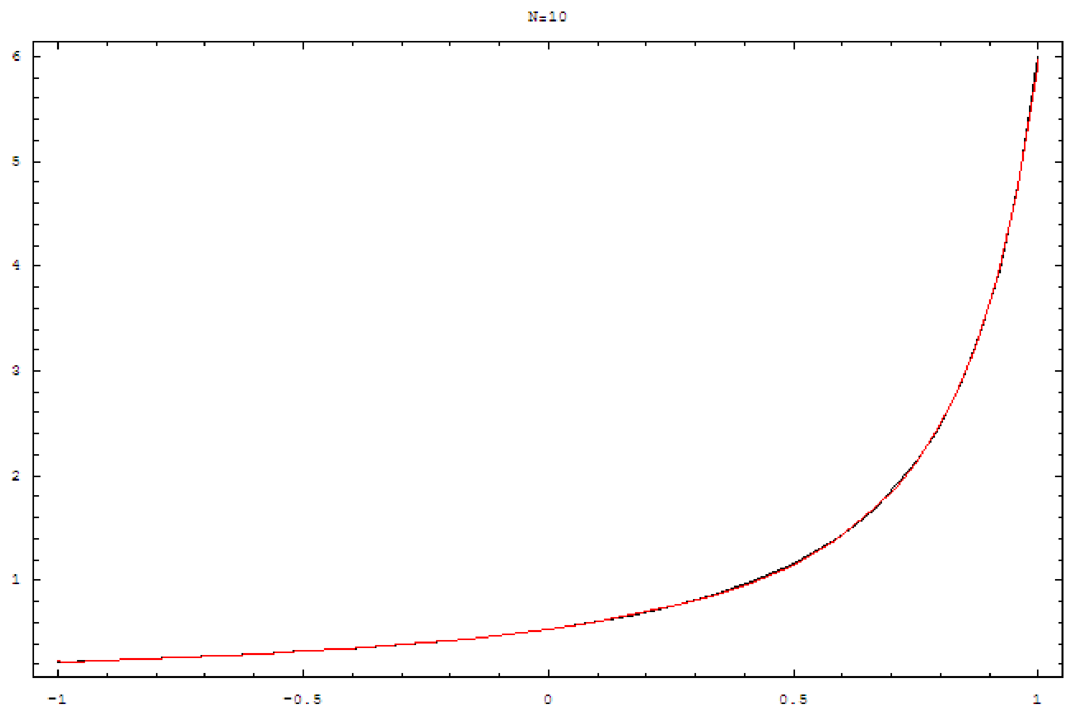
We can continue by including more terms to get desirable accuracy






N=5
6
5
4
3
2
1
0
-1
-0.5
0
0.5
1


N=7
6
5
4
3
2
1
-1
-0.5
0
0.5
1



N=7-10 gives good approximation to the Henyey-Greenstein scattering phase function with g=0.5.

NOTE: The larger the asymmetry parameter g the larger number of terms will be required to achieve acceptable accuracy.