Lecture 9.
Terrestrial infrared radiative processes. Part 2:
Absorption band models.

1. Concept of the equivalent width. Limits of the strong and weak lines.
2. Absorption-band models: Regular (Elsasser) band model and Statistical (Goody) band model.
3. Curtis-Godson Approximation for the inhomogeneous path.

Required reading:
L02: 4.4

1. Concept of the equivalent width. Limits of the strong and weak lines.

Consider a homogeneous atmospheric layer (i.e., the spectral absorption coefficient \( k_{a,v} \) does not change along the path). The spectral transmission function for a band of a width \( \Delta v \) is

\[
T_v(u) = \frac{1}{\Delta v} \int \exp(-k_{a,v}u)dv = \frac{1}{\Delta v} \int \exp(-Sf(v-v_0)u)dv 
\]

and spectral absorptance

\[
A_v(u) = 1 - T_v(u) = \frac{1}{\Delta v} \int (1 - \exp(-k_{a,v}u))dv 
\]

Equivalent width is defined as

\[
W(u) = A_v \Delta v = \int_{\Delta v} [1 - \exp(-k_{a,v}u)]dv \tag{9.1} 
\]

where \( W \) is in units of wavenumber (cm\(^{-1}\)).

- The equivalent width is the width of a fully absorbing (A=1) rectangular-shape line.
Figure 9.1 Schematic illustration of the equivalent width. The dotted rectangular area is equal to the hatched area and represents the total energy absorbed in the line.

**Equivalent width of Lorentz profile:**

Using \( k_{\omega \nu} = S f(\nu - \nu_0) \) and the Lorentz profile of a line, we have

\[
A_\nu(u) = \frac{1}{\Delta \nu} \int_{-\Delta \nu}^{\Delta \nu} (1 - \exp\left( -\frac{S \alpha u}{(\nu - \nu_0)^2 + \alpha^2} \right)) d\nu
\]  

[9.2]

This integral can be expressed in term of the Ladendurg and Reiche function, L(x), as

\[
W = A_\nu \Delta \nu = 2\pi \alpha L(x)
\]  

[9.3]

where \( x = S u/2\pi \alpha \),

S is the line intensity, and u is the absorber amount.

**NOTE:** The Ladendurg and Reiche function L(x) in Eq.[9.3] is given by the modified Bessel functions of the first kind of order \( n \): \( L(x) = x \exp(-x)[I_0(x) + I_1(x)] \), where

\[
I_n(x) = i^{-n} J_n(ix) \quad \text{and} \quad J_n(x) = \frac{i^{-n}}{\pi} \int_0^{\pi} \cos(n\theta) \exp(ix \cos(n\theta)) d\theta
\]

For small \( x \): \( L(x) \) is linear with its asymptotic expansion: \( L(x) = x[1-\ldots] \)

For large \( x \): \( L(x) \) is proportional to a square root of \( x \): \( L(x) = (2x/\pi)^{1/2}[1-\ldots] \)
Case of weak line absorption: either \( k_a \), or \( u \) is small \( \Rightarrow k_a, u \ll 1 \)

Using the asymptotic of \( L(x) \) for small \( x \), we have

\[
A_{\nu}(u) = \frac{W}{\Delta \nu} = \frac{2\pi \alpha L(x)}{\Delta \nu} = 2\pi \alpha \frac{Su}{2\pi \alpha \Delta \nu} = \frac{Su}{\Delta \nu}
\]

Thus

\[
A_{\nu}(u) = \frac{Su}{\Delta \nu}
\]

is called the Linear absorption law. \[9.4\]

Case of strong line absorption: \( Su/\pi \alpha \gg 1 \)

Using the asymptotic of \( L(x) \) for large \( x \), we have

\[
A_{\nu}(u) = \frac{W}{\Delta \nu} = \frac{2\pi \alpha L(x)}{\Delta \nu} = 2\pi \alpha \frac{2x}{\pi} / \Delta \nu = 2\pi \alpha \frac{2Su}{\pi 2\pi \alpha} / \Delta \nu = 2\sqrt{Su} \alpha / \Delta \nu
\]

Thus

\[
A_{\nu}(u) = 2\frac{\sqrt{Su} \alpha}{\Delta \nu}
\]

is called Square root absorption law. \[9.5\]

2. Absorption band models.

The band is defined as a spectral interval of a width \( \Delta \nu \) which is small enough to utilize a mean value of the Plank function \( B_{\nu}(T) \), but large enough so it consists of several absorption lines.

Let’s consider a band with several lines. Two broad cases can be identified:
1) lines have regular positions
2) lines have random positions.

Two main types of band models: regular band models and random band models.
Regular Elsasser band model consists of an infinite array of Lorentz lines of equal intensity, spaced at equal intervals.

EXAMPLE: This type of bands is similar to P and Q branches of linear molecules (e.g., spectrum of N\textsubscript{2}O in the 7.78 \textmu m band; spectrum of CO\textsubscript{2} in the 15 \textmu m band).

The absorption coefficient of the Elsasser bands is

\[ k_{a,v} = \sum_{n=-\infty}^{\infty} \frac{S}{\pi} \frac{\alpha}{(\nu - n\delta)^2 + \alpha^2} \]  

[9.6]

where \( \delta \) is the line spacing (i.e., the distance in wavenumber domain (cm\textsuperscript{-1}) between the centers of two nearest lines).

Figure 9.2 Schematic depiction of the absorption coefficient in the Elsasser (regular) band model, for three different values of \( y = \alpha/\delta \).

NOTE: The parameter of \( y = \alpha/\delta \) can be regarded as a “grayness parameter”: if \( y \) is large, then adjacent lines strongly overlap, so that line structure is increasingly obscured; for small \( y \), the lines are well separated.
Using Eq.[9.6], one can calculate the spectral absorptance as (see derivation in L02 pp.139-141)

\[ A_{\nu} = \text{erf} \left( \frac{\sqrt{\pi} \alpha u \nu}{\Delta} \right) \]  \[ \text{[9.7]} \]

where \( \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} \exp(-x^2)dx \). Values of \( \text{erf}(x) \) are available from standard mathematical tables.

**Principle of statistical (random) models:**
Many spectral bands have random line positions. To approximate this type of bands, various statistical models have been developed.

**EXAMPLE:** The H2O 6.3 µm vibrational-rotational band and H2O rotational band are characterized by random line positions.

**Assumptions:** \( n \) randomly spaced lines with the mean distance \( \Delta \), so that \( \Delta v = n \delta \); lines are independent and have identical shapes, probability density of the strength of \( i \)th line is \( p(S_i) \). Different \( p(S) \) give different models, for instance, Goody, Malkmus, etc.

**Strategy:** derive mean transmission by multiplying transmission of each line at a particular \( \nu \), and integrating over probability distributions of line positions \( \nu_i \) and line strength \( S_i \) for each line.

\[ T_{\nu} = \frac{1}{(\Delta \nu)^n} \int_{\Delta \nu} d\nu_1 \ldots \int_{\Delta \nu} d\nu_n \int_{0}^{\infty} p(S_1) \exp(-uS_1 f(\nu - \nu_{0,1}))dS_1 \ldots \]

\[ \ldots \int_{0}^{\infty} p(S_n) \exp(-uS_n f(\nu - \nu_{0,n}))dS_n = \]

\[ = \prod_{i=1}^{n} \frac{1}{\Delta \nu} \int_{\Delta \nu} d\nu_i \int_{0}^{\infty} p(S_i) \exp(-uS_i f(\nu - \nu_{0,i}))dS_i \]
NOTE: Above equation uses that if lines in a band are uncorrelated, the multiplication law works for average transmittance:

\[ T_{\nu,1,2} = T_{\nu,1} T_{\nu,2} \]

Since in the above equation all integrals alike, we have

\[ T_{\nu} = \left\{ \frac{1}{(\Delta \nu)} \int_{\Delta \nu} d\nu \int_{0}^{\infty} p(S) \exp(-uSf(\nu) dS) \right\}^n = \]

\[ = \{1 - \frac{1}{\Delta \nu} \int_{\Delta \nu} d\nu \int_{0}^{\infty} p(S)[1 - \exp(-uSf(\nu) dS)]^n \] \[ [9.8] \]

The **mean equivalent width** can be defined as

\[ \bar{W} = \int_{0}^{\infty} p(S) \int_{\Delta \nu} [1 - \exp(-uSf(\nu)] d\nu dS \] \[ [9.9] \]

Recalling that \( \Delta \nu = n\delta \), Eq.[9.8] can be rewritten in terms of the **mean equivalent width** giving the mean transmission as

\[ T_{\nu} = \left( 1 - \frac{1}{n} \left( \frac{\bar{W}}{\delta} \right) \right)^n \] \[ [9.10] \]

Since \( \lim_{n \to \infty} \left( 1 - \frac{x}{n} \right)^n \to \exp(-x) \), we have

\[ T_{\nu} = \exp\left( -\frac{\bar{W}}{\delta} \right) \] \[ [9.11] \]

**NOTE**: Single line transmission is \( 1-W/\Delta \nu \), but for many random lines it is exponential in the mean equivalent width.
**Statistical (Goody) band model:**
Consider a band consisting of randomly distributed Lorentz lines.
Assuming that the probability distribution of intensities is the Poisson distribution

\[
p(S) = \bar{S}^{-1} \exp(-S / \bar{S})
\]  

[9.12]

where the \( \bar{S} \) is the mean intensity.

\[
\bar{S} = \int_{0}^{\infty} S p(S) dS
\]

For the Lorentz profile with the mean half-width \( \alpha \), the spectral transmittance can be expressed as

\[
T_{\nu} = \exp \left( -\frac{\bar{S} u}{\delta} \left(1 + \frac{\bar{S} u}{\pi \alpha} \right)^{-1/2} \right)
\]  

[9.13]

Thus, Eq.[9.13] gives the mean spectral transmittance for the Goody random model as a function of path length, \( u \), and two parameters \( \frac{\bar{S}}{\delta} \) and \( \frac{\bar{S}}{\alpha \pi} \).

**Malkmus model:** (has a higher probability of weak lines)
assumes that the probability distribution of intensities is

\[
p(S) = S^{-1} \exp(-S / \bar{S})
\]

and, for a Lorentz line shape, the mean transmittance is

\[
T_{\nu} = \exp \left( -\frac{\pi \alpha}{2 \delta} \left(1 + \frac{4 \bar{S} u}{\pi \alpha} \right)^{1/2} - 1 \right)
\]  

[9.14]

**Weak line limit:**
For \( \frac{\bar{S} u}{\pi \alpha} << 1 \), Eq.[9.13] gives
\[ T_\nu = \exp \left( -\frac{\overline{S}\nu}{\delta} \right) \] \hspace{1cm} [9.15]

**Strong line limit:**

For \( \frac{\overline{S}\nu}{\pi\alpha} \gg 1 \), Eqs.[9.13] and [9.14] give

\[ T_\nu = \exp \left( -\frac{\sqrt{\pi\alpha\overline{S}\nu}}{\delta} \right) \] \hspace{1cm} [9.16]

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**Figure 9.3** Comparison of the Elsasser (solid) and random-Malkmus (dashed) band models for several values of \( y = \alpha/\delta \) (labeled on curves). For both models, curved approach the gray limit (Beer’s law) when \( y \gg 1 \).
3. Curtis-Godson Approximation for inhomogeneous path.

All discussion above was for homogeneous path because band parameters are for one value of pressure and temperature. In the real atmosphere with varying T and P adjustments of the band models are needed to account for the inhomogeneous path

\[ \tau = \int k_{\alpha,\nu}(p(u), T(u)) du \]

**Strategy:** reduce the radiative transfer problem to that of homogeneous path with some sort of averaged values of \( u^*, T^* \) and \( p^* \), so that optical depth can be computed accurately.

**One-parameter scaling approximation:**

Find an equivalent path \( u^* \) at fixed reference temperature \( T_r \) and pressure \( p_r \) that results in the band model having the correct transmission.

Match optical depth for line wings (centers saturated):

\[
\sum_i u^* S_i(T) \alpha_i(p_r, T_r) \pi (\nu - \nu_{o,i})^2 = \int_u u S_i(T) \alpha(p, T) (\nu - \nu_{o,i})^2 du
\]

Re-writing the half-width, \( \alpha \), as

\[ \alpha(P, T) = \alpha(p_r, T_r) \frac{P}{P_r} \left( \frac{T_r}{T} \right)^n \]

We have

\[ u^* = \int_u \left( \frac{p}{p_r} \right)^n \left( \frac{T_r}{T} \right) \rho_a ds \quad [9.17] \]

and thus

\[ \tau_{\nu} = k_{\alpha,\nu}(p_r, T_r) u^* \quad [9.18] \]
Two-parameter scaling approximation (Curtis-Godson approximation):
More accurate band transmission is obtained with the two-parameter approximation.
Want to find optical depth as
\[
\tau = \int_{u} k_{\nu}(p, T) du = k_{a,\nu}(p^*, T^*) u \tag{9.19}
\]
Using Lorentz profile, we have
\[
k_{a,\nu}(p^*, T^*) = \sum_{i} \tilde{S}_{i} \tilde{f}_{\nu, i} = \sum_{i} \frac{\tilde{S}_{i}}{\pi} \frac{\tilde{\alpha}_{i}}{(\nu - \nu_{0,i})^2 + \tilde{\alpha}_{i}^2}
\]
and, thus, two-adjusted parameter \(\tilde{S}\) and \(\tilde{\alpha}\).
They can be introduced as
\[
\tilde{S} = \int_{0}^{u} \tilde{S}(T) du / u \quad \tilde{\alpha} = \int_{0}^{u} \tilde{S}(T) \alpha(p, T) du / \int_{0}^{u} \tilde{S}(T) du
\]