**Lecture 19.**

# Methods for solving the radiative transfer equation with multiple scattering. Part 1: Two-stream approximations.

***Objectives:***

1. Concepts of the reflection and transmission of diffuse radiation by an atmospheric layer.
2. Two-stream approximations.
3. Eddington approximation.
4. Delta-function scaling.

***Required reading***:

L02: 6.3.1, 6.5

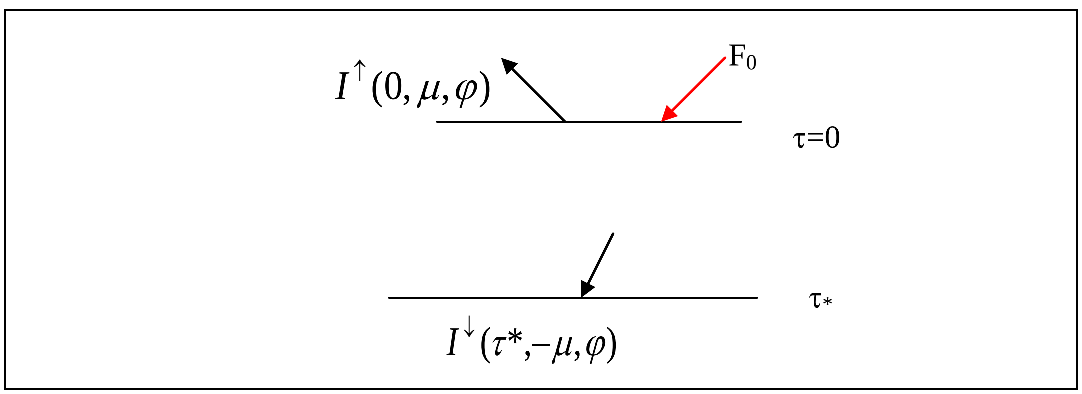
***Additional reading***:

Joseph, J.H., W. J. Wiscombe, and J. A. Weinman, The Delta-Eddington approximation for radiative flux transfer, J. Atmos. Sci. 33, 2452-2459, 1976.

Meador, W.E., and W. R. Weaver, Two stream approximations to radiative transfer in planetary atmospheres: A unified description of existing methods and a new improvement. J.Atmos.Sci. 37, 630-643, 1980.

## 1. Concepts of the reflection and transmission of diffuse radiation by an atmospheric layer.

Consider an atmosphere layer with optical depth $\tau_*$




$I^{\uparrow}(0, \mu, \varphi)$ can be considered as the reflected diffuse intensity

$I^{\downarrow}(\tau^*, -\mu, \varphi)$ can be considered as transmitted diffuse intensity

**Reflection function** of an atmospheric layer is defined as

$$R(\mu,\varphi,\mu_0,\varphi_0) = \pi I^{\uparrow}(0,\mu,\varphi) / \mu_0 F_0 \qquad [19.1]$$

**Transmission function** of an atmospheric layer is defined as

$$T(\mu,\varphi,\mu_0,\varphi_0) = \pi I^{\downarrow}(\tau^*,-\mu,\varphi) / \mu_0 F_0 \qquad [19.2]$$

**NOTE:** Eq.[19.2] uses the diffuse intensity, therefore $T(\mu,\varphi,\mu_0,\varphi_0)$ is also called the diffuse transmission function.

**Transmission function** for direct solar radiation is defined as

$$T_{dir}(\mu_0,\varphi_0) = I_{dir}(\tau^*,-\mu_0,\varphi_0) / \mu_0 F_0 = \exp(-\tau^* / \mu_0) \qquad [19.3]$$

**Planetary albedo (or local albedo or reflection)** is associated with the reflected (upward) flux and defined as

$$r(\mu_0) = \frac{F_{dif}^{\uparrow}(0)}{\mu_0 F_0} = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 R(\mu,\varphi,\mu_0,\varphi_0)\mu d\mu d\varphi \qquad [19.4]$$

**Diffuse transmission** (or transmittance or transmissivity) is associated with transmitted (downward) flux and defined as

$$t(\mu_0) = \frac{F_{dif}^{\downarrow}(\tau^*)}{\mu_0 F_0} = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 T(\mu,\varphi,\mu_0,\varphi_0)\mu d\mu d\varphi \qquad [19.5]$$

For the azimuthally independent case, Eqs.[19.4]-[19.5] reduce to

$$r(\mu_0) = 2\int_0^1 R(\mu,\mu_0)\mu d\mu \qquad [19.6]$$

$$t(\mu_0) = 2\int_0^1 T(\mu,\mu_0)\mu d\mu \qquad [19.7]$$

Consider a planet of radius *a*. The total amount of energy per unit time is

$$\pi a^2 F_0$$

**Spherical (or global) albedo** is a ratio of the energy reflected by the entire planet to the energy incident on it and defined as

$$\bar{r} = \frac{f^{\uparrow}(0)}{\pi a^2 F_0} = 2\int_0^1 r(\mu_0)\mu_0 d\mu_0 \qquad [19.8]$$

**Global diffuse transmission** is defined as

$$\bar{t} = \frac{f^{\downarrow}(\tau_1)}{\pi a^2 F_0} = 2\int_0^1 t(\mu_0)\mu_0 d\mu_0 \qquad [19.9]$$

## 2. Two-stream approximations.

***Underlying idea:***

To find the diffuse radiance, one must solve Eq.[19.5]. Because radiation fluxes and heating rates are angular-averaged properties, one can expect that details of the angular variation of the intensity are not very important for the predictions of these quantities.

***Strategy***:

Introduce an "effective" angular averaged intensity (stream). But one must decide on how to determine the "effective" intensity (i.e., the effective scattering angle $\bar{\mu}^{\uparrow\downarrow}$ ).

***Advantages of the two-stream approximations:***

Two-stream approximations are computationally efficient (therefore they are often used in climate models) and often sufficiently accurate.

***Disadvantages of the two-stream approximations***:

Two-stream methods provide acceptable accuracy but over a restricted range of the parameters. There is no a priory method to estimate the accuracy, so one needs to use the "exact" method to obtain an accurate solution which can be used to estimate the accuracy of two-stream solutions.

***Possible strategies to define the effective scattering angle:***

***i***) define $\overline{\mu}^{\uparrow\downarrow}$ as the intensity-weighted angular means

$$\overline{\mu}^{\uparrow\downarrow} = \frac{\int_0^1 I^{\uparrow\downarrow}(\tau,\mu)\mu d\mu}{\int_0^1 I^{\uparrow\downarrow}(\tau,\mu) d\mu} \qquad [19.10]$$

For **isotropic radiation filed**, Eq.[19.10] gives $\overline{\mu}^{\uparrow\downarrow} = 1/2$

***ii)*** define $\overline{\mu}^{\uparrow\downarrow}$ as the root-mean square value

$$\overline{\mu}^{\uparrow\downarrow} = \sqrt{\langle \mu^2 \rangle} = \sqrt{\frac{\int_0^1 I^{\uparrow\downarrow}(\tau,\mu)\mu^2 d\mu}{\int_0^1 I^{\uparrow\downarrow}(\tau,\mu) d\mu}} \qquad [19.11]$$

For **isotropic radiation filed**, Eq.[19.11] gives $\overline{\mu}^{\uparrow\downarrow} = 1/\sqrt{3}$

**NOTE**: A problem with the above two approaches (Eqs.[19.10] and [19.11]) is that we don't know a priori the angular distribution on the intensity.

**A better strategy is to utilize the Gaussian quadratures**

Gaussian quadrature applied to any function ***f*(μ)** gives

$$\int_{-1}^{1} f(\mu)d\mu \approx \sum_{j=-n}^{n} a_j f(\mu_j) \qquad [19.12]$$

where $a_j$ are the weights defined as

$$a_j = \frac{1}{P_{2n}'(\mu_j)} \int_{-1}^{1} \frac{P_{2n}(\mu)}{\mu - \mu_j} d\mu \qquad [19.13]$$

and $\mu_j$ are the zeros of the even-order Legendre polynomials $P_{2n}(\mu)$, and the prime denotes the differentiation with respect to $\mu_j$.

**NOTE**: Table 6.1 in L02 lists Gaussian points $\mu_j$ and weights $a_j$ for n = 1, 2, 3 and 4.

Recall the equation of the radiative transfer for the diffuse intensity Eq.[19.7] for the azimuth-independent case (see Eq.[19.20]):

$$\mu \frac{dI(\tau,\mu)}{d\tau} = I(\tau,\mu) - \frac{\omega_0}{2}\sum_{l=0}^{N} \varpi_l^* P_l\ (\mu)\int_{-1}^{1} P_l\ (\mu')I(\tau,\mu')d\mu' -$$
$$-\frac{\omega_0}{4\pi}\sum_{l=0}^{N}\varpi_l^* P_l\ (\mu)P_l\ (-\mu_o)F_0 \exp(-\tau/\mu_0)$$

Using Gaussian quadratures, we can re-write this equation as

$$\mu_i \frac{dI(\tau,\mu_i)}{d\tau} = I(\tau,\mu_i) - \frac{\omega_0}{2}\sum_{l=0}^{N} \varpi_l^* P_l\ (\mu_i)\sum_{j=-n}^{n} a_j P_l(\mu_j)I(\tau,\mu_j) -$$
$$-\frac{\omega_0}{4\pi}[\sum_{l=0}^{N}(-1)^l \varpi_l^* P_l\ (\mu_i)P_l\ (-\mu_o)]F_0 \exp(-\tau/\mu_0)$$

[19.14]

where ***i* = *-n, n*** and $\mu_i$***(-n, n)*** represent the directions of radiation streams.

**In the two-stream approximation**, one takes only two streams (i.e., j =-1 and 1) and N=1. Note in table 6.1 in L02 that $\mu_1 = \frac{1}{\sqrt{3}}$ and $a_1 = a_{-1} = 1$

For this case, Eq.[19.14] splits into two equations

$$\mu_1 \frac{dI^{\uparrow}(\tau,\mu_1)}{d\tau} = I^{\uparrow}(\tau,\mu_1) - \omega_0(1-b)I^{\uparrow}(\tau,\mu_1) - \omega_0 bI^{\downarrow}(\tau,-\mu_1) - S^{-}\exp(-\tau/\mu_0)$$

[19.15a]

$$-\mu_1 \frac{dI^{\downarrow}(\tau,-\mu_1)}{d\tau} = I^{\downarrow}(\tau,-\mu_1) - \omega_0(1-b)I^{\downarrow}(\tau,-\mu_1) - \omega_0 bI^{\uparrow}(\tau,\mu_1) - S^{+}\exp(-\tau/\mu_0)$$

[19.15b]

where

$$S^{\pm} = \frac{F_0\omega_0}{4}(1 \pm 3g\mu_1\mu_0)$$

$g = \frac{\varpi_1^*}{3} = \frac{1}{2}\int_{-1}^{1} P(\cos\Theta)\cos(\Theta)d\cos(\Theta)$ , ***g*** is the asymmetry parameter.

$b = \frac{1-g}{2} = \frac{1}{2}\int_{-1}^{1} P(\cos\Theta)\frac{1-\cos(\Theta)}{2}d\cos(\Theta)$ ; ***b*** can be interpreted as a backscattered fraction of energy and ***(1-b)*** is forward scattered energy.

The solutions of Eqs.[19.15a,b] are (see L02,pp.305-306)

$$I^{\uparrow} = I(\tau, \mu_1) = Kv\exp(k\tau) + Hu\exp(-k\tau) + \varepsilon\exp(-\tau/\mu_0) \qquad [19.16a]$$

$$I^{\downarrow} = I(\tau, -\mu_1) = Ku\exp(k\tau) + Hv\exp(-k\tau) + \gamma\exp(-\tau/\mu_0) \qquad [19.16b]$$

where

$$v = (1+a)/2; \quad u = (1-a)/2; \; a^2 = \frac{1-\omega_0}{1-g\omega_0}; \quad k^2 = \frac{(1-\omega_0)(1-g\omega_0)}{\mu_1^2}$$

$$\varepsilon = \frac{\alpha+\beta}{2}; \quad \gamma = \frac{\alpha-\beta}{2}; \quad \alpha = \frac{Z_1\mu_0^2}{1-\mu_0^2 k^2}; \quad \beta = \frac{Z_2\mu_0^2}{1-\mu_0^2 k^2}$$

$$Z_1 = -\frac{(1-g\omega_0)(S^- + S^+)}{\mu_1^2} + \frac{S^- + S^+}{\mu_1\mu_0}; \quad Z_2 = -\frac{(1-\omega_0)(S^- - S^+)}{\mu_1^2} + \frac{S^- + S^+}{\mu_1\mu_0}$$

The constant ***K*** and ***H*** are determined from the boundary conditions at the top and at the bottom of the atmospheric layer. Using the boundary conditions given by Eq.[19.10] (i.e., no diffuse downward radiation at the top of the atmosphere and no reflection from the surface), we have

$$K = -\frac{\varepsilon v\exp(-\tau^*/\mu_0) - \gamma u\exp(-k\tau^*)}{v^2\exp(k\tau^*) - u^2\exp(-k\tau^*)}$$

$$H = -\frac{\varepsilon u\exp(-\tau^*/\mu_0) - \gamma v\exp(-k\tau^*)}{v^2\exp(k\tau^*) - u^2\exp(-k\tau^*)}$$

From the upward and downward intensities we can write expressions for **upward and downward diffuse fluxes** in the two-stream approximations:

$$F^{\uparrow}(\tau) = 2\pi\mu_1 I^{\uparrow}(\tau, \mu_1) \qquad [19.17a]$$

$$F^{\downarrow}(\tau) = 2\pi\mu_1 I^{\downarrow}(\tau, -\mu_1) \qquad [19.17b]$$

## 3. Eddington approximation.

Azimuthally averaged diffuse intensity and scattering phase function may be expanded in terms of Legendre polynomials as

$$I(\tau,\mu)=\sum_{l=0}^{N} I_l(\tau)P_l(\mu)$$

$$P(\mu,\mu')=\sum_{l=0}^{N} \varpi_l^{*} P_l(\mu)P_l(\mu')$$

Note that $P_0(\mu)=1$ and $P_1(\mu)=\mu$.

***Strategy of the Eddington approximation:***

Approximate the radiance field and scattering phase function to first order in μ.

From the above equations,

$$\boldsymbol{I(\tau,\mu)=I_0(\tau)+I_1(\tau)\mu\,;\quad -1\le\mu\le 1} \qquad [19.18]$$

$$P(\mu,\mu')=1+3g\mu\mu' \qquad [19.19]$$

Put Eqs.[19.18] and [19.19] into the azimuthally averaged radiative transfer equation (Eq.[19. 7]), we have

$$\mu\frac{d(I_0+I_1\mu)}{d\tau}=(I_0+I_1\mu)-\frac{\omega_0}{2}\int_{-1}^{1}(I_0+I_1\mu)(1+3g\mu\mu')d\mu'-\frac{\omega_0}{4\pi}F_0(1-3g\mu\mu_0)\exp(-\tau/\mu_0)$$

[19.20]

Taking the integral in Eq.[19.20] results in

$$\mu\frac{d(I_0+I_1\mu)}{d\tau}=(I_0+I_1\mu)-\omega_0(I_0+I_1g\mu)-\frac{\omega_0}{4\pi}F_0(1-3g\mu\mu_0)\exp(-\tau/\mu_0) \qquad [19.21]$$

Rearranging terms gives

$$\mu\frac{dI_0}{d\tau}+\mu^2\frac{dI_1}{d\tau}=I_0(1-\omega_0)+I_1(1-g\omega_0)\mu-\frac{\omega_0}{4\pi}F_0(1-3g\mu\mu_0)\exp(-\tau/\mu_0) \qquad [19.22]$$

First integrate over μ from –1 to 1 and then multiply by μ and integrate from –1 to 1 to get two coupled equations:

$$\frac{dI_1}{d\tau}=3(1-\omega_0)I_0-\frac{3}{4\pi}\omega_0F_0\exp(-\tau/\mu_o) \qquad [19.23a]$$

$$\frac{dI_0}{d\tau}=3(1-\omega_0 g)I_1+\frac{3}{4\pi}\omega_0 g\mu_0F_0\exp(-\tau/\mu_o) \qquad [19.23b]$$

Differentiate Eq.[19.23b] by τ and substitute in Eq.[19.23a]

$$\frac{d^2 I_0}{d\tau^2} = k^2 I_0 - \frac{3}{4\pi}\omega_0(1+g-g\omega_0)F_0\exp(-\tau/\mu_o) \qquad [19.24]$$

where $k^2 = \frac{(1-\omega_0)(1-g\omega_0)}{\mu_1^2}$ is the eigenvalue.

**NOTE**: Eq.[19.24] is known as the diffusion equation for radiative transfer.

The solution of Eq.[19.24] for $\boldsymbol{I_o}$ is exponential in τ:

$$\boxed{I_0 = K\exp(k\tau) + H\exp(-k\tau) + \Psi\exp(-\tau/\mu_o)} \qquad [19.25]$$

where

$$\Psi = \frac{3}{4\pi}\omega_0 F_0 \frac{1+g(1-\omega_0)}{k^2 - 1/\mu_o^2}$$

and the integration constants ***K*** and ***H*** are to be determined from the boundary conditions. Similarly, the solution for $\boldsymbol{I_1}$ can be determined as

$$I_1 = aK\exp(k\tau) - aH\exp(-k\tau) - \xi\exp(-\tau/\mu_o) \qquad [19.26]$$

where $\boldsymbol{a^2 = 3(1-\omega_0)(1-\omega_0 g)}$

$$\xi = \frac{3}{4\pi}\omega_0 \frac{F_0}{\mu_0}\frac{1+3g(1-\omega_0)\mu_0^2}{k^2 - 1/\mu_o^2}$$

Thus <u>diffuse</u> fluxes in the Eddington approximation are

$$F^{\uparrow}(\tau) = 2\pi\int_0^1 [I_0(\tau)+\mu I_1(\tau)]\mu d\mu = \pi\left[I_0(\tau)+\frac{2}{3}I_1(\tau)\right] \qquad [19.27a]$$

$$F^{\downarrow}(\tau) = 2\pi\int_0^{-1} [I_0(\tau)+\mu I_1(\tau)]\mu d\mu = \pi\left[I_0(\tau)-\frac{2}{3}I_1(\tau)\right] \qquad [19.27b]$$

- ✓ The two-stream and Eddington methods are good approximations for optically thick layer, but they may produce inaccurate results for the thin layer and strong absorption cases. The main problem is that the phase function is highly peaked in the forward direction.

✓ For the <u>optically thin atmosphere (τ <1)</u>, the albedo and diffuse transmission are

$$r(\mu_0) = \omega_0 (1/2 - 3g\mu_0/4)\tau^*/\mu_o \qquad [19.28]$$

$$t(\mu_0) = 1 - r - (1-\omega_0)\tau^*/\mu_o \qquad [19.29]$$

Problem: negative reflected flux for $g\mu_0 > 2/3$

➢ ***Example of Eddington solution results***

Consider a uniform layer of optical depth $\tau^*$. In the Eddington approximation for conservative scattering ($\omega_o$=1), the **albedo** (fractional reflected flux) of the layer is

$$r(\mu_0) = \frac{(1-g)\tau^* + (2/3 - \mu_0)(1 - \exp(-\tau^*/\mu_0))}{4/3 + (1-g)\tau^*} \qquad [19.30]$$

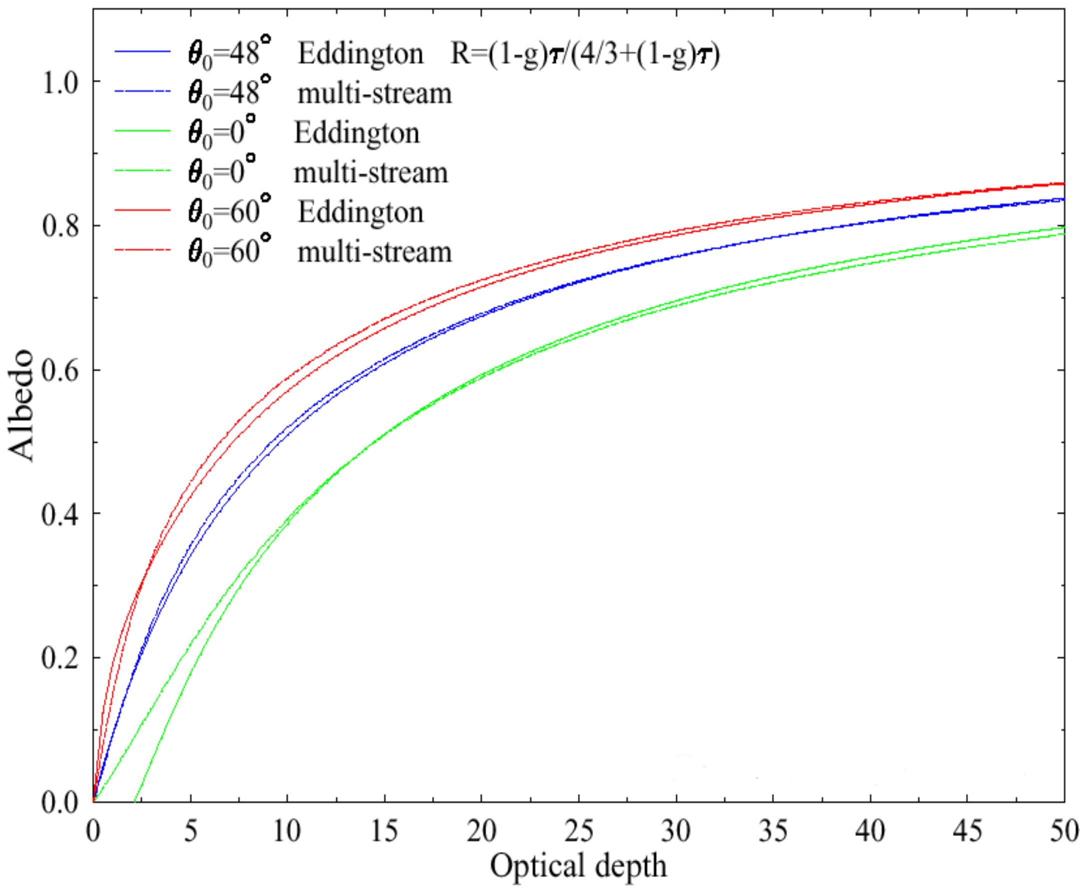



**Figure 19.1** Comparison of Eddington and multistream planetary albedo for conservative scattering ($\lambda$= 0.65 μm, $r_e$ = 10 μm, g = 0.862, i.e. cloud albedo)

***Some properties of reflectivity:***

- ✓ Linear for $\tau$ <<1 and saturation for $\tau$ >>1
- ✓ More forward scattering means less reflection ( $g \uparrow \Rightarrow r \downarrow$ )
- ✓ Higher solar zenith angle means more reflection unless optically thin ( $\mu_0 \downarrow \Rightarrow r \uparrow$ )
- ✓ Multiple scattering amplifies absorption ($\mu_o$= 2/3; g =0.85)

$\tau$ =1, $\omega_o$= 0.99 => r = 0.096, A = 0.019

$\tau$ =10, $\omega_o$= 0.99 => r = 0.45, A = 0.19

$\tau$ =100, $\omega_o$= 0.99 => r = 0.55, A = 0.45

$\tau$ =10, $\omega_o$= 1 => r = 0.92, A = 0.

## 4. Delta-function scaling.

Scattering by atmospheric particulates has the forward diffraction peak and therefore two-term expansion of the scattering phase function (as was done above) is not adequate.

**Delta-function adjustment** replaces a highly peaked phase function with:

(1) a delta function in the forward direction

(2) a smoother scaled phase function (P')

Delta scaling of phase function with **forward scattering fraction f**:

$$P(\cos\Theta) = 2f\delta(1-\cos\Theta) + (1-f)P'(\cos\Theta) \qquad [19.31]$$

Thus the asymmetry parameter is

$$g = \frac{1}{2}\int_{-1}^{1} P(\cos\Theta)\cos\Theta d\cos\Theta = f + (1-f)g' \qquad [19.32]$$

where g' is the scaled asymmetry parameter.

The scaled scattering and absorption optical depth must be

$$\tau_s' = (1-f)\tau_s \text{ and } \tau_a' = \tau_a$$

- **Delta-function adjustment** is introduced to incorporate the forward peak contribution by adjusting optical properties such that the fraction of scattered energy in the forward direction, **f,** is removed from the scattering parameters

$$g' = \frac{g - f}{1 - f} \qquad \omega_o' = \frac{(1 - f)\omega_0}{1 - f\omega_0} \qquad \tau' = (1 - f\omega_0)\tau$$

- The incorporation of the δ-function adjustment into two-stream and Eddington methods greatly improves their accuracy.

***<u>How to get the delta scaling fraction f</u>***

**Delta-isotropic**: make scaled phase function isotropic g'= 0 => f = g

**Delta-Eddington**: make two term scaled function: choose $f = \frac{\varpi_2}{5}$

For instance, for the Henyey-Greenstein phase function: $f = g^2$

thus $g' = \frac{g}{1 + g}$ ; $\omega_o' = \frac{(1 - g^2)\omega_0}{1 - g^2\omega_0}$ ; $\tau' = (1 - g^2\omega_0)\tau$