Lecture 9.

Terrestrial infrared radiative processes. Part 2:

Absorption band models.

1. Concept of the equivalent width. Limits of the strong and weak lines.
2. Absorption-band models: Regular (Elsasser) band model and Statistical (Goody) band model.
3. Curtis-Godson Approximation for the inhomogeneous path.

Required reading:

L02: 4.4

1. Concept of the equivalent width. Limits of the strong and weak lines.

Consider **a homogeneous atmospheric layer** (i.e., the spectral absorption coefficient $\boldsymbol{k_{a,\nu}}$ does not change along the path). The **spectral transmission function** for a band of a width $\Delta\nu$ is

$$T_{\bar{\nu}}(u) = \frac{1}{\Delta\nu}\int_{\Delta\nu} \exp(-k_{a,\nu}u)d\nu = \frac{1}{\Delta\nu}\int_{\Delta\nu} \exp(-Sf(\nu-\nu_0)u)d\nu$$

and **spectral absorptance**

$$A_{\bar{\nu}}(u) = 1 - T_{\bar{\nu}}(u) = \frac{1}{\Delta\nu}\int_{\Delta\nu} (1-\exp(-k_{a,\nu}u))d\nu$$

Equivalent width is defined as

$$W(u) = A_{\bar{\nu}}\Delta\nu = \int_{\Delta\nu} [1-\exp(-k_{a,\nu}u)]d\nu \qquad [9.1]$$

where W is in units of wavenumber (cm^{-1}).

- The **equivalent width** is the width of a fully absorbing (A=1) rectangular-shape line.

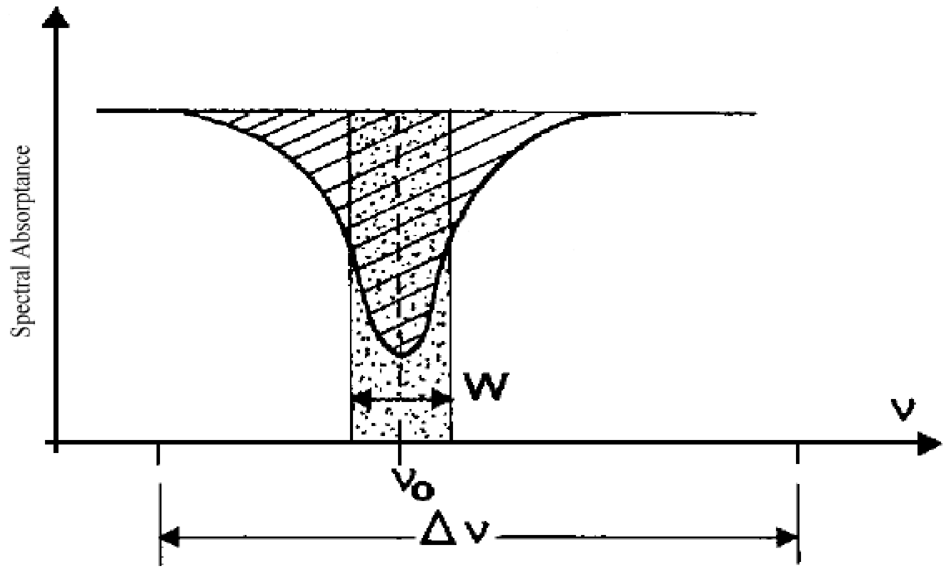

Figure 9.1 Schematic illustration of the equivalent width. The dotted rectangular area is equal to the hatched area and represents the total energy absorbed in the line.

Equivalent width of Lorentz profile:

Using $k_{a,\nu} = S\, f(\nu - \nu_0)$ and the Lorentz profile of a line, we have

$$A_{\bar{\nu}}(u) = \frac{1}{\Delta\nu}\int_{\Delta\nu}(1-\exp\left(-\frac{S\alpha u/\pi}{(\nu-\nu_0)^2+\alpha^2}\right))d\nu \qquad [9.2]$$

This integral can be expressed in term of the Ladendurg and Reiche function, L(x), as

$$\boxed{W = A_{\bar{\nu}}\Delta\nu = 2\pi\alpha L(x)} \qquad [9.3]$$

where $\boldsymbol{x = Su/2}\pi\alpha$,

S is the line intensity, and **u** is the absorber amount.

NOTE: The Ladendurg and Reiche function L(x) in Eq.[9.3] is given by the modified Bessel functions of the first kind of order n: $L(x) = x\exp(-x)[I_0(x) + I_1(x)]$, where

$$I_n(x) = i^{-n}J_n(ix) \quad \text{and} \quad J_n(x) = \frac{i^{-n}}{\pi}\int_0^{\pi}\cos(n\theta)\ \exp(ix\ \cos(n\theta))d\theta$$

For small x: L(x) is linear with its asymptotic expansion: L(x)= x[1-…]

For large x: L(x) is proportional to a square root of x: $L(x) = (2x/\pi)^{1/2}[1-\ldots]$

Case of weak line absorption: either $k_{a,\nu}$ or u is small => $\boldsymbol{k_{a,\nu}u \ll 1}$

Using the asymptotic of L(x) for small x, we have

$$A_{\bar{\nu}}(u) = \frac{W}{\Delta\nu} = 2\pi\alpha L(x)/\Delta\nu = 2\pi\alpha \frac{Su}{2\pi\alpha\Delta\nu} = \frac{Su}{\Delta\nu}$$

Thus

$$\boxed{A_{\bar{\nu}}(u) = \frac{Su}{\Delta\nu}}$$ is called the Linear absorption law. [9.4]

Case of strong line absorption: $\boldsymbol{Su/\pi\alpha \gg 1}$

Using the asymptotic of L(x) for large x, we have

$$A_{\bar{\nu}}(u) = \frac{W}{\Delta\nu} = 2\pi\alpha L(x)/\Delta\nu = 2\pi\alpha\sqrt{\frac{2x}{\pi}}/\Delta\nu =$$

$$= 2\pi\alpha\sqrt{\frac{2Su}{\pi 2\pi\alpha}}/\Delta\nu = 2\sqrt{Su\alpha}/\Delta\nu$$

Thus

$$\boxed{A_{\bar{\nu}}(u) = 2\frac{\sqrt{Su\alpha}}{\Delta\nu}}$$ is called Square root absorption law. [9.5]

2. Absorption band models.

The **band** is defined as a spectral interval of a width **Δν** which is small enough to utilize a mean value of the Plank function $B_{\bar{\nu}}(T)$, but large enough so it consists of several **absorption lines**.

Let's consider a band with several lines. Two broad cases can be identified:

1) lines have **regular positions**

2) lines have **random positions**.

⇩

Two main types of band models: **regular** band models and **random** band models.

Regular Elsasser band model consists of an infinite array of Lorentz lines of equal intensity, spaced at equal intervals.

EXAMPLE: This type of bands is similar to P and Q branches of linear molecules (e.g., spectrum of N_2O in the 7.78 μm band; spectrum of CO_2 in the 15 μm band).

The **absorption coefficient** of the Elsasser bands is

$$k_{a,\nu} = \sum_{n=-\infty}^{\infty} \frac{S}{\pi} \; \frac{\alpha}{(\nu - n\delta)^2 + \alpha^2} \qquad [9.6]$$

where δ is the line spacing (i.e., the distance in wavenumber domain (cm^{-1}) between the centers of two nearest lines).

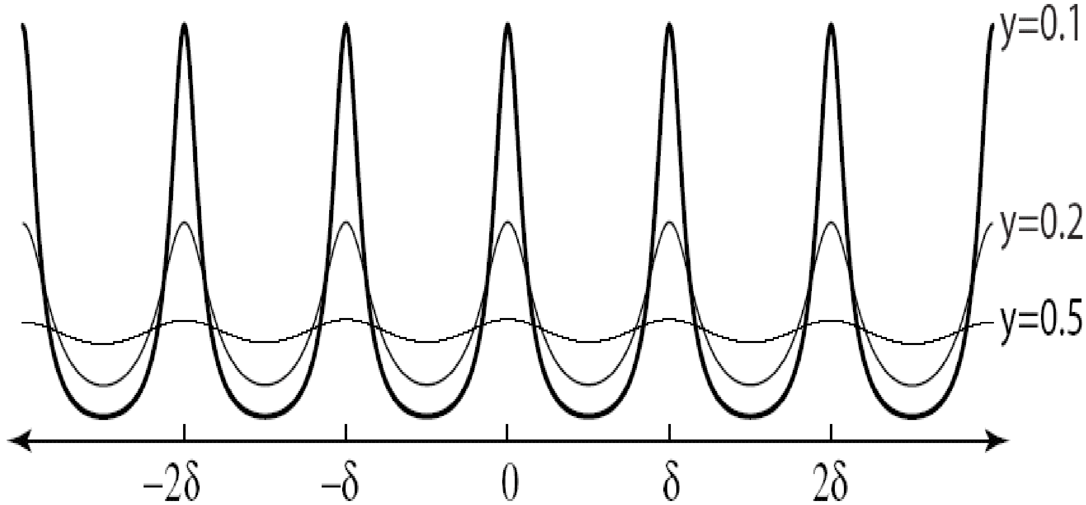

Figure 9.2 Schematic depiction of the absorption coefficient in the Elsasser (regular) band model, for three different values of y= α/δ.

NOTE: The parameter of y= α/δ can be regarded as a "grayness parameter": if y is large, then adjacent lines strongly overlap, so that line structure is increasingly obscured; for small y, the lines are well separated.

Using Eq.[9.6], one can calculate the spectral absorptance as (see derivation in L02 pp.139-141)

$$A_{\bar{\nu}} = erf\left(\frac{\sqrt{\pi S \alpha u}}{\delta}\right) \qquad [9.7]$$

where $erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-x^2)dx$. Values of ***erf(x)*** are available from standard mathematical tables.

<u>Principle of statistical (random) models:</u>

Many spectral bands have random line positions. To approximate this type of bands, various statistical models have been developed.

EXAMPLE: The H_2O 6.3 μm vibrational-rotational band and H_2O rotational band are characterized by random line positions.

<u>Assumptions</u>: ***n*** randomly spaced lines with the mean distance δ, so that Δν= nδ ; lines are independent and have identical shapes, probability density of the strength of ***i***'th line is ***p(Si)***. Different ***p(S)*** give different models, for instance, Goody, Malkmus, etc.

<u>Strategy</u>: derive mean transmission by multiplying transmission of each line at a particular **ν**, and integrating over probability distributions of line positions ν_i and line strength S_i for each line.

$$T_{\bar{\nu}} = \frac{1}{(\Delta\nu)^n}\int_{\Delta\nu} d\nu_1 .... \int_{\Delta\nu} d\nu_n \int_0^\infty p(S_1)\exp(-uS_1 f(\nu-\nu_{0,1}))dS_1 ...$$

$$...\int_0^\infty p(S_n)\exp(-uS_n f(\nu-\nu_{0,n}))dS_n =$$

$$= \prod_{i=1}^{n} \frac{1}{\Delta\nu}\int_{\Delta\nu} d\nu_i \int_0^\infty p(S_i)\exp(-uS_i\, f(\nu-\nu_{0,i}))dS_i$$

NOTE: Above equation uses that if lines in a band are uncorrelated, the multiplication law works for average transmittance:

$$T_{\bar{\nu},1,2} = T_{\bar{\nu},1} T_{\bar{\nu},2}$$

Since in the above equation all integrals alike, we have

$$T_{\bar{\nu}} = \{\frac{1}{(\Delta\nu)} \int_{\Delta\nu} d\nu \int_0^\infty p(S) \exp(-uSf(\nu) dS\}^n =$$
$$= \{1 - \frac{1}{\Delta\nu} \int_{\Delta\nu} d\nu \int_0^\infty p(S)[1 - \exp(-uSf(\nu) dS\}^n \qquad [9.8]$$

The **mean equivalent width** can be defined as

$$\overline{W} = \int_0^\infty p(S) \int_{\Delta\nu} [1 - \exp(-uSf(\nu)] d\nu dS \qquad [9.9]$$

Recalling that Δν= nδ , Eq.[9.8] can be rewritten in terms of the **mean equivalent width** giving the mean transmission as

$$T_{\bar{\nu}} = \left(1 - \frac{1}{n}\left(\frac{\overline{W}}{\delta}\right)\right)^n \qquad [9.10]$$

Since $\lim_{n->\infty} (1 - \frac{x}{n})^n -> \exp(-x)$, we have

$$\boxed{T_{\bar{\nu}} = \exp(-\frac{\overline{W}}{\delta})} \qquad [9.11]$$

NOTE: Single line transmission is **1-W/Δν,** but for many random lines it is exponential in the mean equivalent width.

Statistical (Goody) band model:

Consider a band consisting of randomly distributed Lorentz lines.

Assuming that the probability distribution of intensities is the **Poisson distribution**

$$p(S) = \bar{S}^{-1} \exp(-S/\bar{S}) \qquad [9.12]$$

where the $\bar{S}$ in the mean intensity.

$$\bar{S} = \int_0^\infty Sp(S)dS$$

For the Lorentz profile with the mean half-width α, the spectral transmittance can be expressed as

$$T_{\bar{\nu}} = \exp\left(-\frac{\bar{S}u}{\delta}\left(1+\frac{\bar{S}u}{\pi\alpha}\right)^{-1/2}\right) \qquad [9.13]$$

Thus, Eq.[9.13] gives the mean spectral transmittance for the Goody random model as a function of path length, ***u***, and two parameters $\frac{\bar{S}}{\delta}$ and $\frac{\bar{S}}{\alpha\pi}$.

Malkmus model: (has a higher probability of weak lines)

assumes that the probability distribution of intensities is

$$p(S) = S^{-1} \exp(-S/\bar{S})$$

and, for a Lorentz line shape, the mean transmittance is

$$T_{\bar{\nu}} = \exp\left(-\frac{\pi\alpha}{2\delta}\left(\left(1+\frac{4\bar{S}u}{\pi\alpha}\right)^{1/2}-1\right)\right) \qquad [9.14]$$

Weak line limit:

For $\frac{\bar{S}u}{\pi\alpha} << 1$, Eq.[9.13] gives

$$T_{\bar{\nu}} = \exp\left(-\frac{\bar{S}u}{\delta}\right) \qquad [9.15]$$

Strong line limit:

For $\frac{\bar{S}u}{\pi\alpha} >> 1$, Eqs.[9.13] and [9.14] give

$$T_{\bar{\nu}} = \exp\left(-\frac{\sqrt{\pi\alpha\bar{S}u}}{\delta}\right) \qquad [9.16]$$

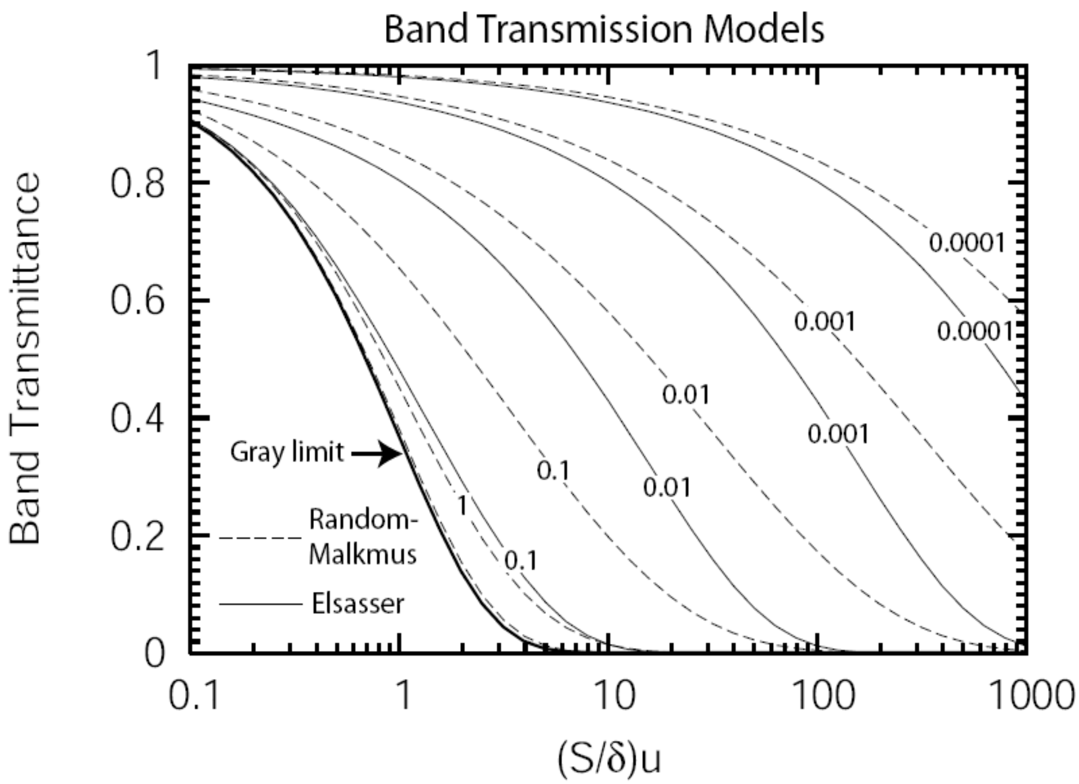

Figure 9.3 Comparison of the Elsasser (solid) and random-Malkmus (dashed) band models for several values of y= α/δ (labeled on curves). For both models, curved approach the gray limit (Beer's law) when y >> 1.

3. Curtis-Godson Approximation for inhomogeneous path.

All discussion above was for homogeneous path because band parameters are for one value of pressure and temperature. In the real atmosphere with varying T and P adjustments of the band models are needed to account for the **inhomogeneous path**

$$\tau = \int_u k_{a,\nu}(p(u),T(u))du$$

Strategy: reduce the radiative transfer problem to that of homogeneous path with some sort of averaged values of u*, T* and p*, so that optical depth can be computed accurately.

One-parameter scaling approximation:

Find an equivalent path u* at fixed reference temperature $\boldsymbol{T_r}$ and pressure $\boldsymbol{p_r}$ that results in the band model having the correct transmission.

Match optical depth for line wings (centers saturated):

$$\sum_i \frac{u^* S_i(T)\alpha_i(p_r,T_r)}{\pi(\nu-\nu_{o,i})^2} = \int_u \sum_i \frac{uS_i(T)\alpha(p,T)}{(\nu-\nu_{o,i})^2}du$$

Re-writing the half-width, α, as

$$\alpha(P,T) = \alpha(p_r,T_r)\frac{P}{P_r}\left(\frac{T_r}{T}\right)^n$$

We have

$$u^* = \int_u \left(\frac{p}{p_r}\right)\left(\frac{T_r}{T}\right)^n \rho_a ds \qquad [9.17]$$

and thus

$$\boxed{\tau_\nu = k_{a,\nu}(p_r,T_r)u^*} \qquad [9.18]$$

Two-parameter scaling approximation (Curtis-Godson approximation):

More accurate band transmission is obtained with the two-parameter approximation.

Want to find optical depth as

$$\tau = \int_u k_\nu(p,T)du = k_{a,\nu}(p^*,T^*)u \qquad [9.19]$$

Using Lorentz profile, we have

$$k_{a,\nu}(p^*,T^*) = \sum_i \tilde{S}_i \tilde{f}_{\nu,i} = \sum_i \frac{\tilde{S}_i}{\pi} \frac{\tilde{\alpha}_i}{(\nu-\nu_{0,i})^2 + \tilde{\alpha}_i^2}$$

and, thus, two-adjusted parameter $\tilde{S}$ and $\tilde{\alpha}$.

They can be introduced as

$$\tilde{\bar{S}} = \int_0^u \bar{S}(T)du / u \qquad \tilde{\alpha} = \int_0^u \bar{S}(T)\alpha(p,T)du / \int_0^u \bar{S}(T)du$$