Lecture 19.

Methods for solving the radiative transfer equation with scattering. Part 1: Two-stream approximations.

1. Concepts of the reflection and transmission of diffuse radiation by an atmospheric layer.

2. Two-stream approximations.

3. Eddington approximation.

4. Delta-function scaling.

Required reading:

L02: 6.3.1, 6.5

Additional reading (posted at class website):

Joseph, J.H., W. J. Wiscombe, and J. A. Weinman, The Delta-Eddington approximation for radiative flux transfer, J. Atmos. Sci. 33, 2452-2459, 1976.

Meador, W.E., and W. R. Weaver, Two stream approximations to radiative transfer in planetary atmospheres: A unified description of existing methods and a new improvement. J.Atmos.Sci. 37, 630-643, 1980.

Concepts of reflection and transmission of an atmospheric layer:

Reflection function of an atmospheric layer is defined as

$$R(\mu,\varphi,\mu_0,\varphi_0) = \pi I^{\uparrow}(0,\mu,\varphi)/\mu_0 F_0$$

Transmission function of an atmospheric layer is defined as

$$T(\mu,\varphi,\mu_0,\varphi_0) = \pi I^{\downarrow}(\tau^*,-\mu,\varphi)/\mu_0 F_0$$

Planetary albedo (or local albedo or reflection) is associated with the reflected (upward) flux and defined as

$$r(\mu_0) = \frac{F_{dif}^{\uparrow}(0)}{\mu_0 F_0} = \frac{1}{\pi}\int_0^{2\pi}\int_0^{1} R(\mu,\varphi,\mu_0,\varphi_0)\mu d\mu d\varphi$$

Diffuse transmission is associated with transmitted (downward) flux and defined as

$$t(\mu_0) = \frac{F_{dif}^{\downarrow}(\tau^*)}{\mu_0 F_0} = \frac{1}{\pi}\int_0^{2\pi}\int_0^{1} T(\mu,\varphi,\mu_0,\varphi_0)\mu d\mu d\varphi$$

Spherical (or global) albedo is a ratio of the energy reflected by the entire planet to the energy incident on it and defined as

$$\bar{r} = \frac{f^{\uparrow}(0)}{\pi a^2 F_0} = 2\int_0^1 r(\mu_0)\mu_0 d\mu_0$$

Global diffuse transmission is defined as

$$\bar{t} = \frac{f^{\downarrow}(\tau^*)}{\pi a^2 F_0} = 2\int_0^1 t(\mu_0)\mu_0 d\mu_0$$

Two-stream approximations.

Underlying concept:

Because radiation flux and heating rates are angular-averaged properties, one can expect that details of the angular variation of intensity are not very important for the predictions of these quantities.

Strategy:

Introduce an "effective" angular averaged intensity (stream). But one must decide on how to determine the "effective" intensity (i.e., the effective scattering angle $\bar{\mu}^{\uparrow\downarrow}$).

Advantages:

Two-stream approximations are computationally efficient (therefore they are often used in climate and NWP models) and often sufficiently accurate.

Disadvantages:

- Two-stream methods provide acceptable accuracy but over a restricted range of the parameters.
- There is no a priory method to estimate the accuracy, so one needs to use the "exact" method to obtain an accurate solution which can be used to estimate the accuracy of two-stream solutions.

Possible strategies to define the effective scattering angle:

i) define as the intensity-weighted angular means

$$\overline{\mu}^{\uparrow\downarrow} = \frac{\int_0^1 I^{\uparrow\downarrow}(\tau,\mu)\mu d\mu}{\int_0^1 I^{\uparrow\downarrow}(\tau,\mu)d\mu}$$

ii) define as the root-mean square value

$$\overline{\mu}^{\uparrow\downarrow} = \sqrt{\left\langle \mu^2 \right\rangle} = \sqrt{\frac{\int_0^1 I^{\uparrow\downarrow}(\tau,\mu)\mu^2 d\mu}{\int_0^1 I^{\uparrow\downarrow}(\tau,\mu)d\mu}}$$

Problem: do not know the angular distribution of intensity!!!

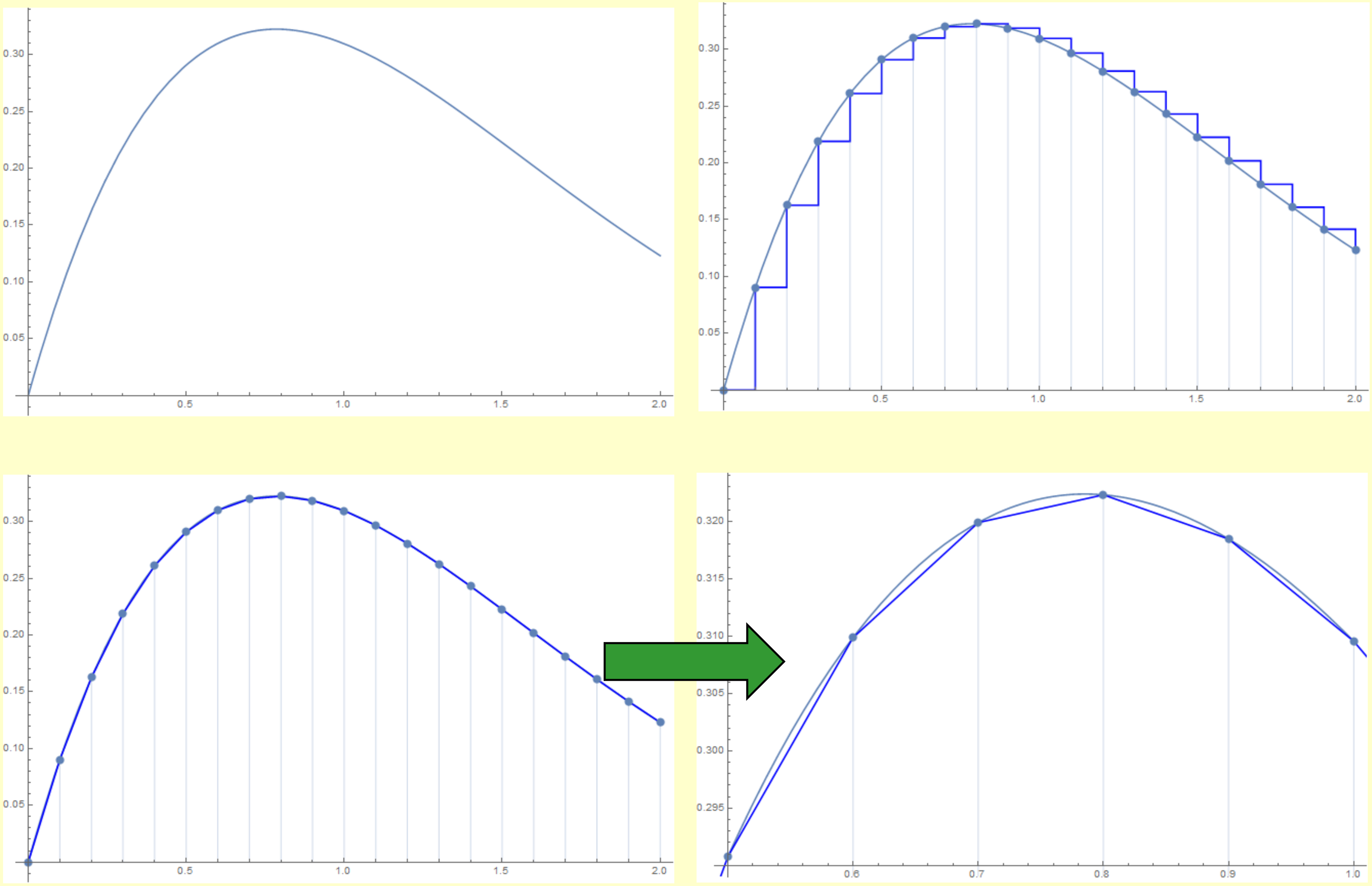
A better strategy: use Gaussian quadratures

Gaussian quadratures applied to any function *f*(μ) give

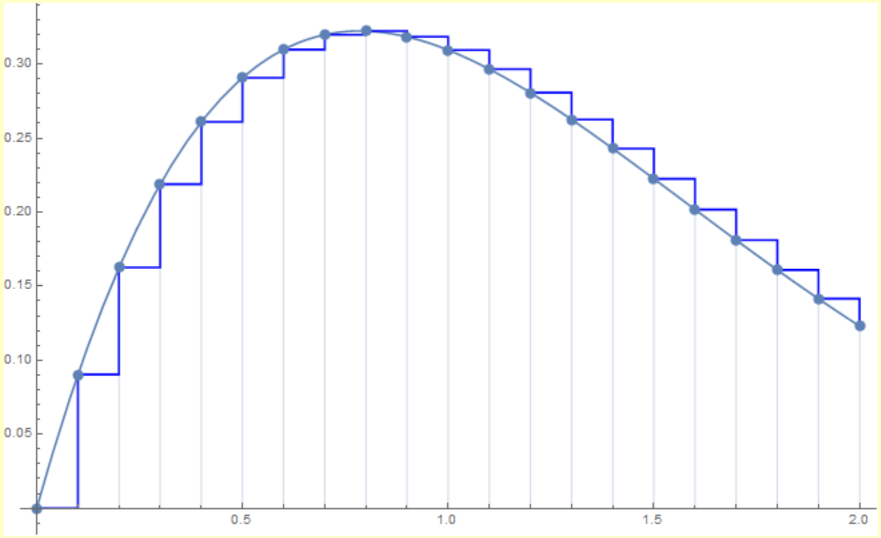
$$\int_{-1}^{1} f(\mu)d\mu \approx \sum_{j=-n}^{n} a_j f(\mu_j)$$

$$a_j = \frac{1}{P_{2n}{}'(\mu_j)} \int_{-1}^{1} \frac{P_{2n}(\mu)}{\mu - \mu_j} d\mu$$

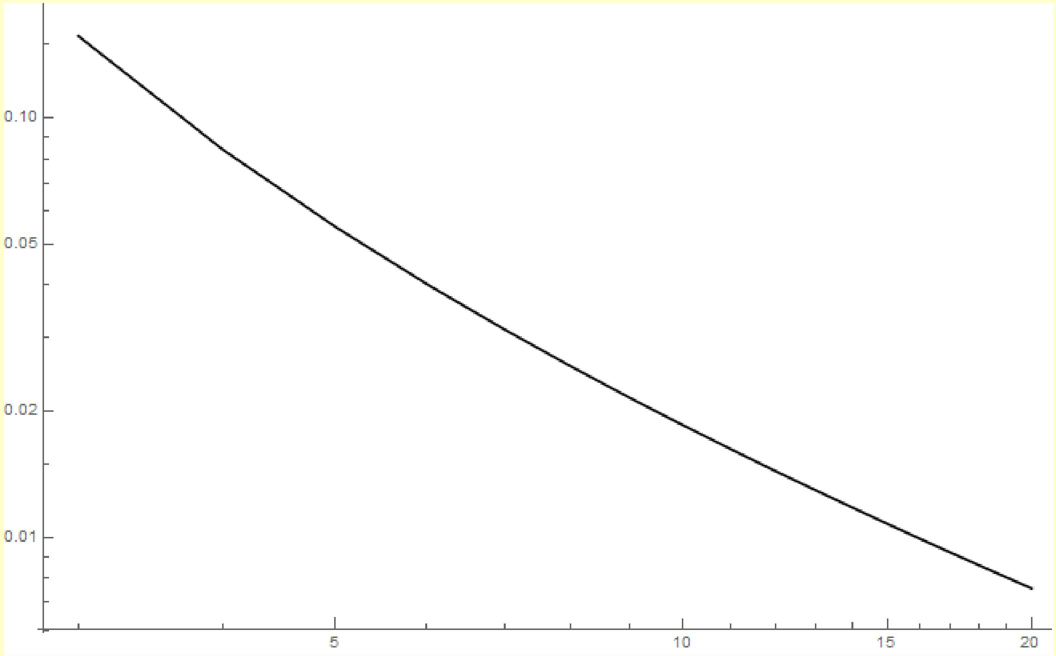
How to calculate Integrals



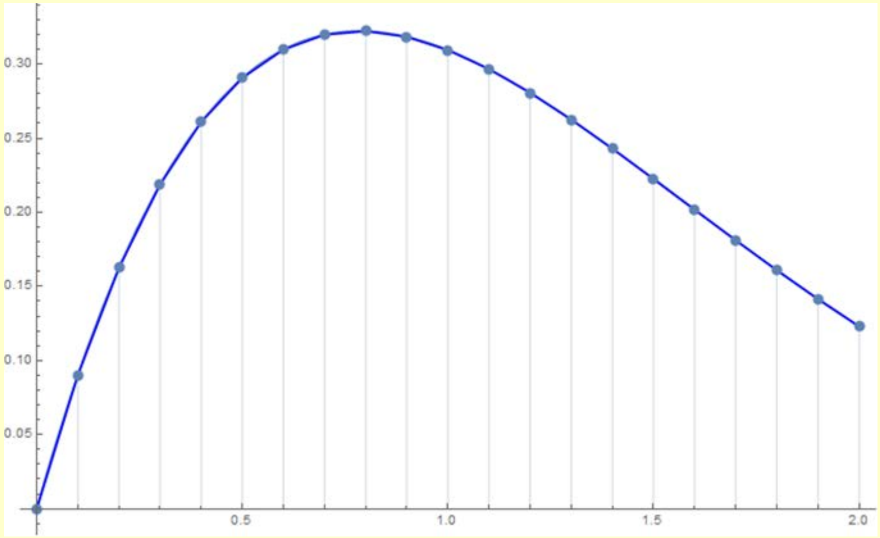
How to calculate Integrals



$$\int_a^b f(x)dx = \Delta x \sum_{k=1}^{N-1} f(x_k)$$



How to calculate Integrals



$$\int_a^b f(x)dx = \Delta x \sum_{k=1}^{N-1} (f(x_{k+1}) + f(x_k))/2$$

How to calculate Integrals?

$$\int_a^b f(x)dx = \sum_{k=1}^{N} w_k f(x_k)$$

There are N free parameters: w_k, so the error $\delta \sim \Delta x^N$

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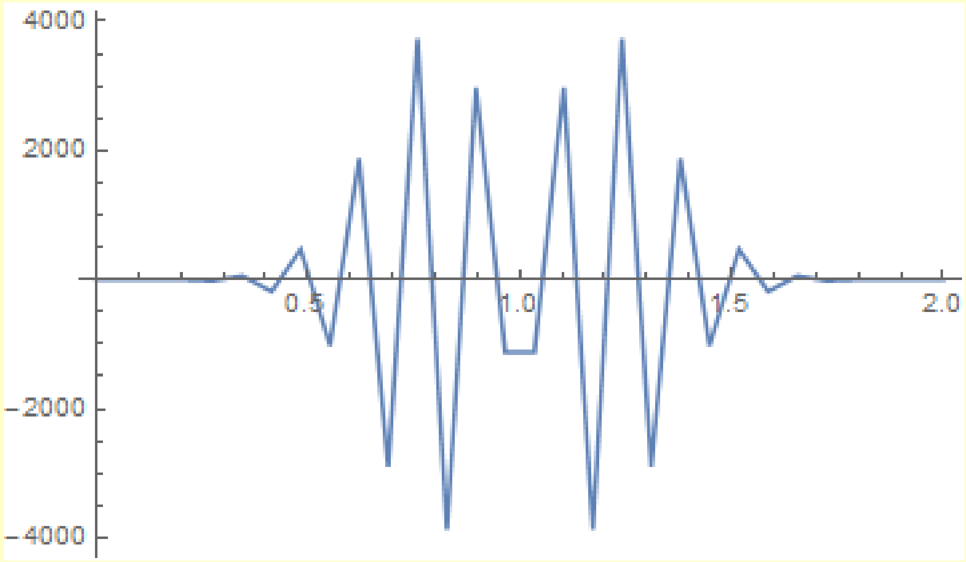

$$\sum_{k=1}^{N} w_k = (b-a)$$

How to calculate Integrals?

$$\int_a^b f(x)dx = \sum_{k=1}^{N} w_k f(x_k)$$

We can choose not only w, but x also !!!
Instead of N free parameters we'll have 2N

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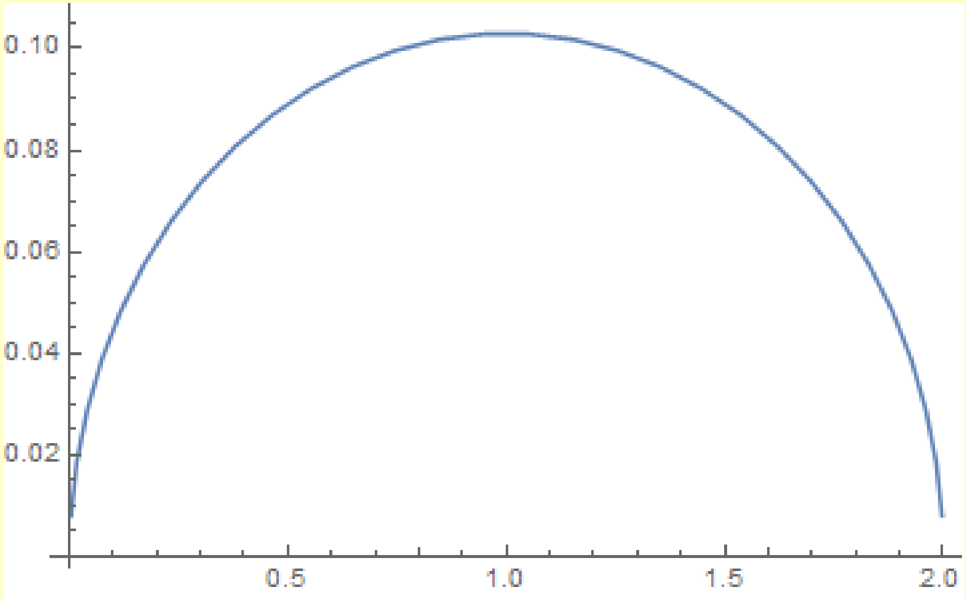

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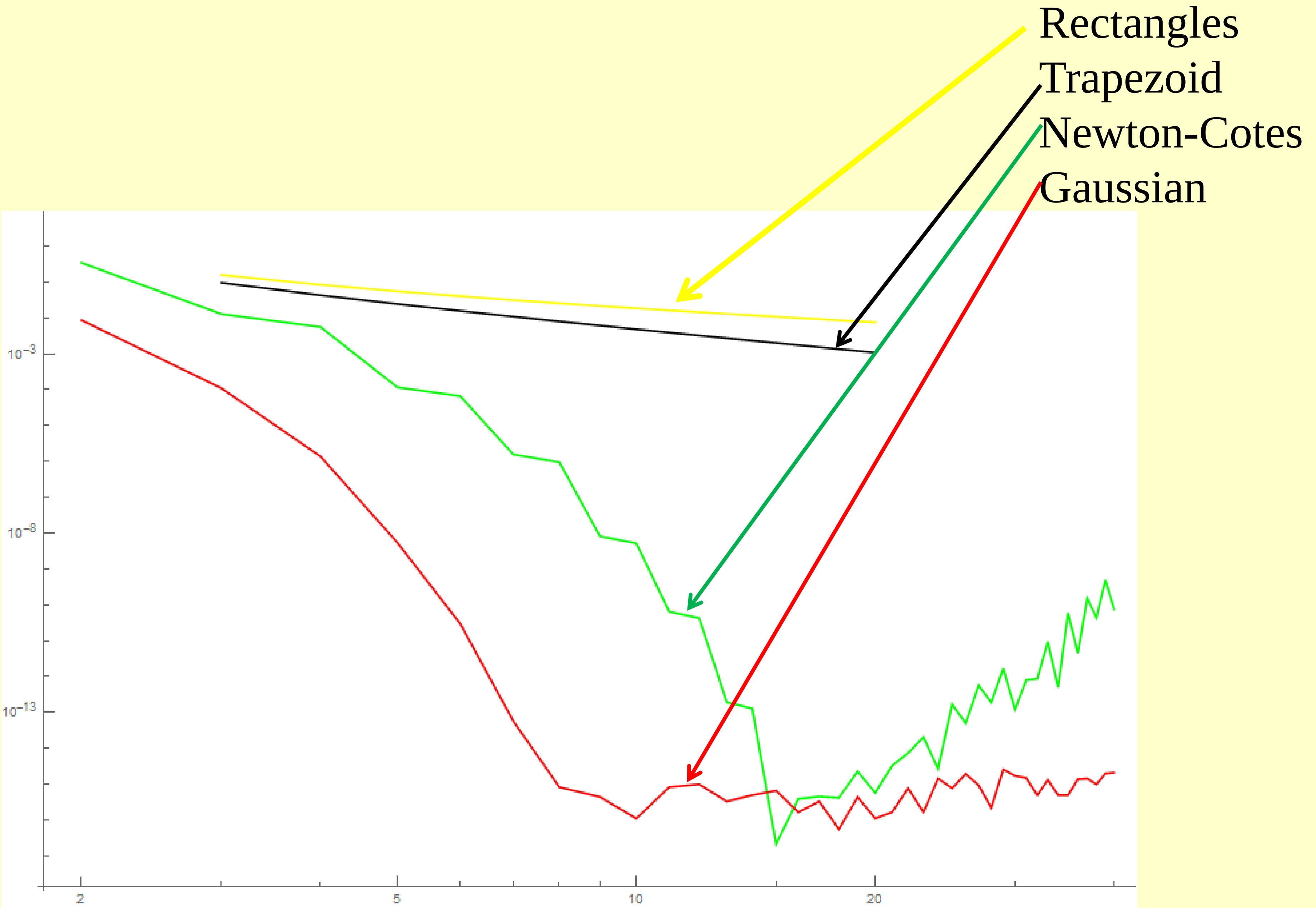


How to calculate Integrals?

How to choose optimal x_i and w_i?

$$\int_a^b f(x)\rho(x)dx = \sum_{k=1}^{N} w_k f(x_k)$$

$$\varphi(x) = (x - x_1)(x - x_2)\ldots(x - x_N)$$

$$\int_a^b x^0 \varphi(x)\rho(x)dx = 0$$

$$\int_a^b x^1 \varphi(x)\rho(x)dx = 0$$

$$\int_a^b P_k(x)P_m(x)\rho(x)dx = \delta_{k,m}$$

$$\int_a^b x^{N-1} \varphi(x)\rho(x)dx = 0$$

How to calculate Integrals?

How to choose optimal x_i and w_i?

$$\int_a^b f(x)\rho(x)dx = \sum_{k=1}^{N} w_k f(x_k)$$

$$\int_a^b x^k \rho(x)dx = \sum_{i=1}^{N} w_i x_i^k = I(k), \quad k = 0,\ldots N-1$$

$a = -1$
$b = +1$
$\rho(x) = 1$

$$\Rightarrow \quad w_k = \frac{2}{1 - {x_k}^2}[P_n'(x_k)]^2$$

$a = -1$
$b = +1$
$\rho(x) = 1/\sqrt{1 - x^2}$

$$\Rightarrow \quad x_k = cos\left(\frac{2k-1}{2n}\pi\right) \quad w_k = \frac{\pi}{n}$$

Azimuthally independent case

$$P(\mu,\mu') = \frac{1}{2\pi}\int_0^{2\pi} P(\mu,\varphi,\mu',\varphi')\,d\varphi'$$

$$\mu\frac{dI(\tau,\mu)}{d\tau} = I(\tau,\mu) - \frac{\omega_0}{2}\int_{-1}^{1} I(\tau,\mu')P(\mu,\mu')\,d\mu' - \frac{\omega_0}{4\pi}F_0 P(\mu,-\mu_0)\exp(-\tau/\mu_0)$$

Expand the phase function $$P(\mu,\mu') = \sum_{l=0}^{N} \varpi_l P_l(\mu) P_l(\mu')$$

$$\mu \frac{dI(\tau,\mu)}{d\tau} = I(\tau,\mu) - \frac{\omega_0}{2}\sum_{l=0}^{N} \varpi_l^* P_l(\mu)\int_{-1}^{1} P_l(\mu')I(\tau,\mu')d\mu' -$$

$$-\frac{\omega_0}{4\pi}\sum_{l=0}^{N}\varpi_l^* P_l(\mu)P_l(-\mu_o)F_0 \exp(-\tau/\mu_0)$$

See eq.[18.20]

Using Gaussian quadratures, re-write the above equation:

$$\mu_i \frac{dI(\tau,\mu_i)}{d\tau} = I(\tau,\mu_i) - \frac{\omega_0}{2}\sum_{l=0}^{N} \varpi_l^* P_l(\mu_i)\sum_{j=-n}^{n} a_j P_l(\mu_j) I(\tau,\mu_j) -$$

$$-\frac{\omega_0}{4\pi}[\sum_{l=0}^{N}(-1)^l \varpi_l^* P_l(\mu_i)P_l(-\mu_o)]F_0 \exp(-\tau/\mu_0)$$

Gaussian Quadrature

where ***i* = -*n*, *n*** and **μ_i(-*n*, *n*)** represent the directions of radiation streams.

In the two-stream approximation: only two streams (i.e., j =-1 and 1) and N=1. From table 6.1 in L02 that and $a_1 = a_{-1} = 1$

In the two-stream approximation

$$\mu_1 \frac{dI^{\uparrow}(\tau,\mu_1)}{d\tau} = I^{\uparrow}(\tau,\mu_1) - \omega_0(1-b)I^{\uparrow}(\tau,\mu_1) - \omega_0 b I^{\downarrow}(\tau,-\mu_1) - S^{-}\exp(-\tau/\mu_0)$$

$$-\mu_1 \frac{dI^{\downarrow}(\tau,-\mu_1)}{d\tau} = I^{\downarrow}(\tau,-\mu_1) - \omega_0(1-b)I^{\downarrow}(\tau,-\mu_1) - \omega_0 b I^{\uparrow}(\tau,\mu_1) - S^{+}\exp(-\tau/\mu_0)$$

$$S^{\pm} = \frac{F_0\omega_0}{4}(1 \pm 3g\mu_1\mu_0)$$

$$b = \frac{1-g}{2} = \frac{1}{2}\int_{-1}^{1} P(\cos\Theta)\frac{1-\cos(\Theta)}{2}\, d\cos(\Theta)$$

$$g = \frac{\varpi_1^*}{3} = \frac{1}{2}\int_{-1}^{1} P(\cos\Theta)\cos(\Theta)\, d\cos(\Theta)$$

Solutions

$$I^{\uparrow} = I(\tau, \mu_1) = Kv\exp(k\tau) + Hu\exp(-k\tau) + \varepsilon\exp(-\tau/\mu_0)$$

$$I^{\downarrow} = I(\tau, -\mu_1) = Ku\exp(k\tau) + Hv\exp(-k\tau) + \gamma\exp(-\tau/\mu_0)$$

Upward and downward diffuse fluxes in the two-stream approximations:

$$F^{\uparrow}(\tau) = 2\pi\mu_1 I^{\uparrow}(\tau, \mu_1)$$

$$F^{\downarrow}(\tau) = 2\pi\mu_1 I^{\downarrow}(\tau, -\mu_1)$$

μ_1- Gaussian point

Eddington approximation.

$$I(\tau,\mu) = \sum_{l=0}^{N} I_l(\tau) P_l(\mu) \qquad P(\mu,\mu') = \sum_{l=0}^{N} \varpi_l^* P_l(\mu) P_l(\mu')$$

Strategy of the Eddington approximation:

Approximate the radiance field and scattering phase function to first order in μ (i.e., N=1)

$$I(\tau,\mu) = I_0(\tau) + I_1(\tau)\mu$$

$$P(\mu,\mu') = 1 + 3g\mu\mu'$$

$$\mu \frac{dI(\tau,\mu)}{d\tau} = I(\tau,\mu) - \frac{\omega_0}{2}\int_{-1}^{1} I(\tau,\mu')P(\mu,\mu')d\mu' - \frac{\omega_0}{4\pi}F_0 P(\mu,-\mu_0)\exp(-\tau/\mu_0)$$

$$\mu \frac{d(I_0 + I_1\mu)}{d\tau} = (I_0 + I_1\mu) - \frac{\omega_0}{2}\int_{-1}^{1} (I_0 + I_1\mu)(1+3g\mu\mu')d\mu' - \frac{\omega_0}{4\pi}F_0(1-3g\mu\mu_0)\exp(-\tau/\mu_0)$$

$$\frac{dI_1}{d\tau} = 3(1-\omega_0)I_0 - \frac{3}{4\pi}\omega_0 F_0 \exp(-\tau/\mu_o)$$

$$\frac{dI_0}{d\tau} = 3(1-\omega_0 g)I_1 + \frac{3}{4\pi}\omega_0 g\mu_0 F_0 \exp(-\tau/\mu_o)$$

Solutions

$$I_0 = K\exp(k\tau) + H\exp(-k\tau) + \Psi\exp(-\tau/\mu_o)$$

$$\Psi = \frac{3}{4\pi}\omega_0 F_0 \frac{1 + g(1-\omega_0)}{k^2 - 1/\mu_o^2}$$

$$I_1 = aK\exp(k\tau) - aH\exp(-k\tau) - \xi\exp(-\tau/\mu_o)$$

$$\xi = \frac{3}{4\pi}\omega_0 \frac{F_0}{\mu_0} \frac{1 + 3g(1-\omega_0)\mu_0^2}{k^2 - 1/\mu_o^2}$$

$$F^{\uparrow}(\tau) = 2\pi\int_0^1 \left[I_0(\tau) + \mu I_1(\tau)\right]\mu d\mu = \pi\left[I_0(\tau) + \frac{2}{3}I_1(\tau)\right]$$

$$F^{\downarrow}(\tau) = 2\pi\int_0^{-1} \left[I_0(\tau) + \mu I_1(\tau)\right]\mu d\mu = \pi\left[I_0(\tau) - \frac{2}{3}I_1(\tau)\right]$$

• The two-stream and Eddington methods are good approximations for optically thick layer, but they may produce inaccurate results for thin layers and strong absorption. The main problem is that the phase function is highly peaked in the forward direction.

• For the optically thin atmosphere, the albedo and diffuse transmission are

$$r(\mu_0) = \omega_0 (1/2 - 3g\mu_0/4)\tau^*/\mu_o$$

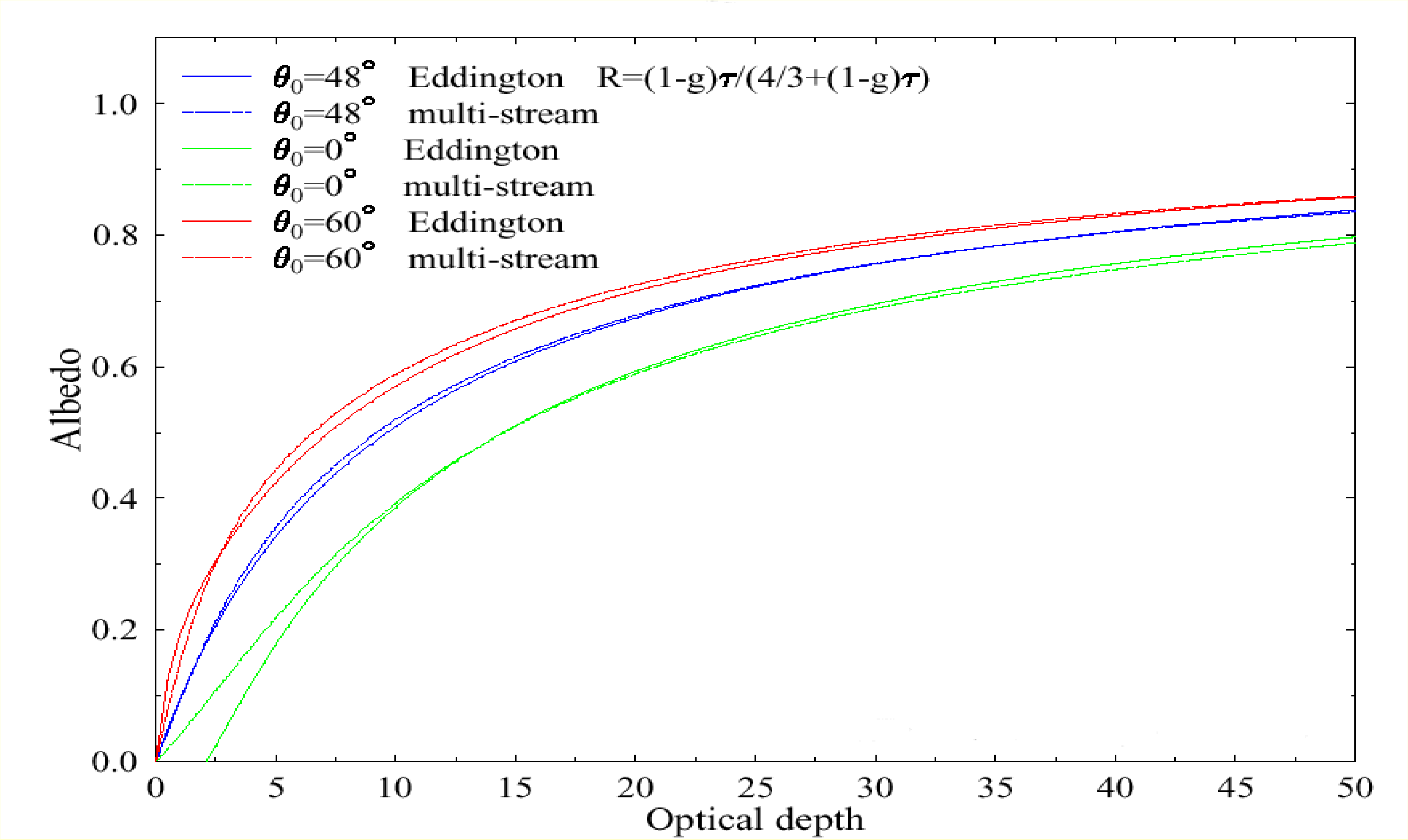
$$t(\mu_0) = 1 - r - (1-\omega_0)\tau^*/\mu_o$$

Problem: negative reflected flux for $g\mu_0 > 2/3$

Example of Eddington solution results

In the Eddington approximation for conservative scattering (ω_o=1),
the **albedo** (fractional reflected flux) of the layer is

$$r(\mu_o) = \frac{(1-g)\tau^* + (2/3 - \mu_o)(1 - \exp(-\tau^*/\mu_o))}{4/3 + (1-g)\tau^*}$$



For what optical depth does the Eddington optically thin limit albedo deviate by 10% from the conservative (no absorption) scattering Eddington albedo? Use a solar angle of μ_0=2/3 and assume the asymmetry parameter g=0.95. What would this thin limit optical depth be for isotropic scattering?

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The Eddington optically thin limit albedo is

$$R_{thin} = \omega(1/2 - 3g\mu_0/4)\tau/\mu_0$$

For conservative scattering $\omega = 1$ and for $\mu_0 = 2/3$ the thin limit albedo is

$$R_{thin} = 1/2(1-g)\tau/\mu_0 = 3/4(1-g)\tau$$

For $\mu_0 = 2/3$ the conservative scattering solution is

$$R_{edd} = \frac{(1-g)\tau}{4/3 + (1-g)\tau}$$

The thin limit deviates by 10% when

$$\frac{R_{thin}}{R_{edd}} = 1.1 \qquad \frac{3}{4}\left[\frac{4}{3} + (1-g)\tau\right] = 1.1$$

$$\tau_{thin} = 0.13/(1-g) = 0.95$$

For isotropic scattering ($g = 0$):

$$\tau_{thin} = 0.13/(1-g) = 0.13$$

Delta-function adjustment replaces a highly peaked phase function with:

(1) a delta function in the forward direction

(2) a smoother scaled phase function (P') for side scattering

Delta scaling of phase function with **forward scattering fraction f**:

$$P(\cos\Theta) = 2f\delta(1-\cos\Theta) + (1-f)P'(\cos\Theta)$$

$$g = \frac{1}{2}\int_{-1}^{1} P(\cos\Theta)\cos\Theta d\cos\Theta = f + (1-f)g'$$

$$\tau_s' = (1-f)\tau_s$$

$$\omega_o' = \frac{(1-f)\omega_0}{1-f\omega_0}$$

$$\tau_a' = \tau_a$$