Lecture 9
Terrestrial infrared radiative processes. Part 2:
K-distribution approximation.

Objectives:
2. Correlated k-distribution approximation (CKD).

Required reading:
L02:4.3

Recommended reading:


KD method is developed to compute the spectral transmittance (hence the spectral intensity or spectral fluxes) based on grouping of gaseous absorption coefficients.

NOTE: The k-distribution approach was proposed by Ambartzumian in 30th as an alternative to the computationally expensive line-by-line methods.

- KD method benefits from the fact that the same value of $k_\nu$ is encountered many times over a given spectral interval => thus to eliminate the redundancy, one can group the values of k and perform the transmittance calculation only once for a given value of k.
**Strategy:**
Consider a **homogeneous** atmospheric layer. Spectral transmission is (see Eq.[8.5], Lecture 8)

\[
T_v(u) = \frac{1}{\Delta v} \int \exp(-k_v u) d\nu
\]

In a homogeneous atmospheric layer, spectral transmittance is independent of the ordering of \( k \) in a given spectral range, i.e., the order in which the wavenumbers are summed does not matter => so sum them from low to high \( k \)

Thus, we want to replace the integration over the wavenumber by an integration over \( k \). It can be done by introducing a **normalized probability distribution function** for \( k \)

\[
T_v(u) = \frac{1}{\Delta v} \int \exp(-k_v u) d\nu = \int_0^\infty \exp(-k u)f(k)dk 
\]

[9.1]

where we set \( \int_0^\infty f(k) = 1 \)

\( f(k) \) is the fraction of the spectral band with absorption coefficient \( k \rightarrow k+dk \)

\( f(k) \) is a smooth function

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**Figure 9.1 (a)** A schematic of absorption line spectra at two different pressure. (b) The two probability density function \( f(k) \) associated with (a). The shaded area shows the strongest absorption.
The cumulative probability function can be defined as

\[ g(k) = \int_0^k f(k) \, dk \tag{9.2} \]

and \( g(0) = 0; \ g(\infty) = 1 \) and \( dg(k) = f(k) \, dk \).

\( g(k) \) is the fraction of the spectrum with absorption coefficient below \( k \)

**NOTE:** By definition, \( g(k) \) is a monotonically increasing and smooth function in k-space, Therefore, \( k(g) \), as an inverse function of \( g(k) \), is a smooth function in g-space.

Therefore, the spectral transmittance can be written as

\[ T_{\nu}(u) = \frac{1}{\Delta \nu} \int_{\nu}^{\nu + \Delta \nu} \exp(-k_{\nu}u) \, d\nu = \int_0^\infty \exp(-k_u)f(k) \, dk = \int_0^1 \exp(-k(g)u) \, dg \tag{9.3} \]

**Figure 9.2** (a) Absorption coefficient \( k_{\nu} \) (in cm\(^{-1}\) atm\(^{-1}\)) as a function of wavenumber in 9.6 \( \mu \)m ozone band (resolution of 0.05 cm\(^{-1}\), p=30 mb, T=200K). (b) The probability density function \( f(k) \) of the absorption coefficient. (c) The cumulative probability distribution function as a function of \( k \). (d) Same as (c) but \( k \) vs. \( g \).
Because both $g(k)$ and $k(g)$ are smooth functions, the above integral can be calculated by a finite sum as

$$T_v(u) = \int_0^1 \exp(-k(g)u)dg \approx \sum_{i=1}^N \exp(-k(g_i)u)\Delta g_i =$$

$$= \Delta g_1 e^{-k_1u} + \Delta g_2 e^{-k_2u} + \ldots + \Delta g_N e^{-k_Nu} \quad [9.4]$$

where $\Delta g_i$ is the quadrature weight.

Thus the **KD method** allows calculating the spectral transmittance as a finite weighted sum of exponent in $g$-space, replacing the tedious wavenumber integration.

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**Numerical realization of KD:**

*(see illustration below)*

Consider a spectral interval $\Delta \nu$ that contains numerous absorption lines.

Let’s divide it into $N$ intervals of $\Delta \nu_j$, $j = 1, 2, 3 \ldots N$.

The probability distribution function can be written as

$$f(k) = \frac{1}{\Delta \nu} \frac{d\nu}{dk} = \frac{1}{\Delta \nu} \sum_j \left| \frac{\Delta \nu_j}{\nu_j} \right|$$

where $\Delta \nu_j$ is the subinterval of $\Delta \nu$ where $k$ is a monotonic function of $\nu$.

Then the cumulative probability is

$$g(k) = \frac{1}{\Delta \nu} \sum_j \int_0^k \left| \frac{\Delta \nu_j}{\Delta k} \right| dk' = \frac{1}{\Delta \nu} \sum_j \int_0^k \Delta \nu_j(k) = \frac{n(0,k)}{N}$$

where $n(0,k)$ is the number of computational points that contribute to $k$ cumulatively.
**Figure 9.3** How to calculate the absorption coefficient in g-space from the known absorption coefficient in the wavenumber domain. Solid line gives absorption coefficient as a function of $\nu$. Numbers on the right side are the data points in each $\Delta k$ interval (total number $N=35$).

Thus by definition, $g(j\Delta k) = n(0, j\Delta k)/N$

CKD is the extension of KD for inhomogeneous atmosphere.

Each pressure and temperature along the path has unique $k_\nu$ spectrum.

**CKD method** sorts each $k_\nu(p, T)$ spectrum independently to make k-distributions $k(g, p, T)$ for each $p$ and $T$.

NOTE: In practice, discrete k-distributions $k_j(p_b, T_m)$ are made for a set of pressures $p_b$ and temperatures $T_m$ and interpolated in between to any $p$ and $T$.

- **Overlap of gases in spectral band:**

What do we do about multiple gases absorbing in one spectral band?

**Overlap method #1:** assume that absorption spectra are independent.

$$\int_{\Delta \nu} T_{\nu}^{(1)} T_{\nu}^{(2)} d\nu = T_{\nu}^{(1)} T_{\nu}^{(2)} = \sum_{i=1}^{N} \exp(-k_i(g)u_1) \Delta g_{1,i} \sum_{j}^{M} \exp(-k_j(g)u_2) \Delta g_{2,j}$$

$$T_{\nu}(u_1, u_2) = \sum_{i=1}^{N} \Delta g_{1,i} \sum_{j}^{M} \Delta g_{2,j} \exp(-\tau_{mn})$$

where $\tau_{mn} = k_{1m}u_1 + k_{2n}u_2$, thus we have $M \times N$ terms

**Overlap method #2:** introduce mixing ratio as an additional factor, so $k(g, p, T, q)$
An example of correlated k-distribution


Divides shortwave into 6 bands with total 54 $k$’s and longwave into 12 bands with total of 121 $k$’s.

<table>
<thead>
<tr>
<th>Band</th>
<th>Region (cm$^{-1}$)</th>
<th>$N$’s</th>
<th>Gases</th>
<th>Solar Flux (W/m$^2$)</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>50000-14500</td>
<td>10</td>
<td>O3</td>
<td>619.62</td>
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<tr>
<td>2</td>
<td>14500-7700</td>
<td>8</td>
<td>H2O</td>
<td>484.30</td>
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<td>3</td>
<td>7700-5250</td>
<td>12</td>
<td>H2O</td>
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<td>4</td>
<td>5250-4000</td>
<td>7</td>
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<td>4000-2850</td>
<td>12</td>
<td>H2O</td>
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<td>6</td>
<td>2850-2500</td>
<td>5</td>
<td>H2O</td>
<td>5.80</td>
</tr>
<tr>
<td>7</td>
<td>2200-1900</td>
<td>2</td>
<td>H2O</td>
<td>Overlap method</td>
</tr>
<tr>
<td>8</td>
<td>1900-1700</td>
<td>3</td>
<td>H2O</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>1700-1400</td>
<td>4</td>
<td>H2O</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>1400-1250</td>
<td>4</td>
<td>H2O, CH4, N2O</td>
<td>1</td>
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<tr>
<td>11</td>
<td>1250-1100</td>
<td>3</td>
<td>H2O, CH4, N2O</td>
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<tr>
<td>12</td>
<td>1100-980</td>
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<td>H2O, O3</td>
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<tr>
<td>13</td>
<td>980-800</td>
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</tr>
<tr>
<td>14</td>
<td>800-670</td>
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</tr>
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<td>12</td>
<td>H2O, CO2</td>
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<tr>
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<td>400-280</td>
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<tr>
<td>18</td>
<td>280-0</td>
<td>8</td>
<td>H2O</td>
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</tbody>
</table>

NOTE: A more accurate CKD is available from the AER (Mlawer et al.) It uses 16 $k$’s in each band and has 16 bands in the longwave and 14 bands in the shortwave.